Mould Calculus,
Combinatorial Hopf Algebras,
and
The Jacobian Conjecture.

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Linearization, normal forms of Vectors Fields
Linearization, normal forms of Diffeos
Jacobian Conjecture

↓

Compute some Identity-tangent formal diffeos :
Find their coefficients with the help of
equations involving composition, inverse, ...

↓

Mould calculus machinery

↓

Find some characters on :
Shuffle, Quasishuffle, Connes-Kreimer H. A.
Composition $\longrightarrow$ Convolution (coproduct)
Inverse $\longrightarrow$ Inverse (antipode)

Some facts on the Jacobian Conjecture

JC : ($\nu \geqslant 2$) Let φ a polynomial map from $\mathbb{C}^\nu$ to $\mathbb{C}^\nu$, if $J_\varphi(x)$ is constant, non zero ($x = (x_1, \ldots, x_\nu)$), then it is invertible and its inverse ψ is polynomial.

1. By affine change of coordinates, one can focus on the case φ identity-tangent :

$$\varphi(x) = x + f(x) = (x_1 + f_1(x), \ldots, x_\nu + f_\nu(x))$$

2. If f is homogeneous of degree d with $d \geqslant 2$

$$\exists k \geq 2, \ \forall x, \quad [J_f(x)]^k = 0$$

3. We will focus on the Jacobian conjecture $\mathrm{JC}_{k,d}$ with nilpotency k and degree d.
4. In fact (Bass–Connell–Wright, 1982), if $\mathrm{JC}_{k,3}$ is true for all k, then JC is true.
5. $\mathrm{JC}_{k,2}$ is true (Wang, 1980), with no control on the degree of the inverse.
6. $\mathrm{JC}_{3,2}$ is true (Wright, Singer) and

$$\deg(\psi) \leqslant 6$$

with the help of trees ...

7. φ is locally invertible and its inverse ψ is analytic : one can see φ and ψ as formal identity-tangent diffeomorphisms

Formal identity-tangent diffeomorphisms and Substitution automorphisms.

Let $G \subset \mathbb{C}[[x]]^\nu$ be the group (for the composition) of formal identity–tangent diffeomorphisms :

1. G acts on $\mathbb{C}[[x]]$: for φ in G we define the substitution automorphism :

$$\begin{array}{rcccl} F_\varphi & : & \mathbb{C}[[x]] & \to & \mathbb{C}[[x]] \\ & & A(x) & \mapsto & (F_\varphi.A)(x) = A(\varphi(x)) \end{array}$$

   and F_φ is linear, $F_\varphi.(AB) = (F_\varphi.A)(F_\varphi.B)$,

$$F_\varphi.x = (F_\varphi.x_1, \dots, F_\varphi.x_\nu) = \varphi(x) \in G$$

2. Conversely, substitution automorphisms form a group isomorphic to G : $F_\varphi.F_\psi = F_{\psi\circ\varphi}$
3. Taylor Formula : if $\varphi(x) = x + f(x) \in G$,

$$\begin{array}{rcl} F_\varphi.A(x) & = & A(x+f(x)) \\ & = & A(x) + \\ & & \displaystyle\sum_{n\geqslant 1} \frac{1}{n!} \sum_{i_1,\dots,i_n} f_{i_1}(x)\dots f_{i_n}(x)\partial_{x_{i_1}}\dots\partial_{x_{i_n}} A(x) \\ & = & \displaystyle(\mathrm{Id} + \sum_{s\geqslant 1} \mathbb{D}_n).A(x) \end{array}$$

4. If f is homogeneous of degree d ($\mathrm{JC}_{k,d}$), then : if P homogeneous and $\deg(P) = m$ then $\deg(\mathbb{D}_n.P) = m + n(d-1)$ and

$$\mathbb{D}_n(AB) = A(\mathbb{D}_n B) + \sum_{k=1}^{n-1} (\mathbb{D}_k A)(\mathbb{D}_{n-k} B) + (\mathbb{D}_n A) B$$

Here comes the quasishuffle Hopf algebra !

$\mathcal{QSH}$ is the graded Hopf algebra whose linear basis is given by words

$$\boldsymbol{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, ..., n_s),\ n_i \geqslant 1\}$$

Its product is the quasishuffle product :

$$\begin{aligned}\pi((n_1) \otimes (n_2, n_3)) &= (n_1, n_2, n_3) + (n_2, n_1, n_3) + (n_2, n_3, n_1) \\ &\quad + (n_1 + n_2, n_3) + (n_2, n_1 + n_3)\end{aligned}$$

Its coproduct is given by the deconcatenation

$$\Delta \boldsymbol{n} = \sum_{\boldsymbol{n}^1 . \boldsymbol{n}^2 = \boldsymbol{n}} \boldsymbol{n}^1 \otimes \boldsymbol{n}^2$$

Let φ be a given polynomial map in G, then

$$\begin{array}{rccc}\rho_\varphi^{\mathrm{qs}} : & \mathcal{QSH}^* & \to & \mathbb{C}[x, \partial_x] \\ & \boldsymbol{n} = (n_1, ..., n_s) & \mapsto & \mathbb{D}_{n_s} ... \mathbb{D}_{n_1} = \mathbb{D}_{\boldsymbol{n}}\end{array}$$

is an antialgebra morphism ($\rho_\varphi^{\mathrm{qs}}(\emptyset) = \mathbb{D}_\emptyset = \mathrm{Id}_{\mathbb{C}[[x]]}$) such that

$$\rho_\varphi^{\mathrm{qs}}(\boldsymbol{n})(AB) = \pi_{\mathbb{C}[[x]]} \circ ((\rho_\varphi^{\mathrm{qs}} \otimes \rho_\varphi^{\mathrm{qs}}) \circ \Delta(\boldsymbol{n}))(A \otimes B)$$

Now we are ready for some mould calculus.

Mould Calculus

Definition 1. *A symmetrel mould (Ecalle) on* $\boldsymbol{N}$ *with values in* $\mathbb{C}$, $M^\bullet = \{M^{\boldsymbol{n}} \in \mathbb{C}, \boldsymbol{n} \in \boldsymbol{N}\}$ *is the collection of the values of a character* χ *of* $\mathcal{C}(\mathcal{QSH}, \mathbb{C})$ *on its basis* ($M^{\boldsymbol{n}} = \chi(\boldsymbol{n})$)*. We have the product of moulds :*

$$(M_1^\bullet \times M_2^\bullet)^{\boldsymbol{n}} = (\chi_1 * \chi_2)(\boldsymbol{n}) = \pi_{\mathbb{C}} \circ (\chi_1 \otimes \chi_2) \circ \Delta(\boldsymbol{n})$$

Theorem 2. *Let* φ *in* G*, then the map*

$$\begin{array}{rccc} \mathcal{S}_\varphi^{\text{qs}} : & \mathcal{C}(\mathcal{QSH}, \mathbb{C}) & \rightarrow & G \\ & \chi & \mapsto & \displaystyle\sum_{\boldsymbol{n} \in \boldsymbol{N}} \chi(\boldsymbol{n}) \rho_\varphi^{\text{qs}}(\boldsymbol{n}).x \end{array}$$

is a group morphism and the mould series :

$$\sum_{\boldsymbol{n} \in \boldsymbol{N}} \chi(\boldsymbol{n}) \rho_\varphi^{\text{qs}}(\boldsymbol{n}) = \sum_{\boldsymbol{n} \in \boldsymbol{N}} M^{\boldsymbol{n}} \mathbb{D}_{\boldsymbol{n}}$$

is the substitution automorphism associated to $\mathcal{S}_\varphi(\chi)$

Remark 3. Let ξ be the character defined by $\xi(\emptyset) = 1$ and

$$\xi(n_1, \ldots, n_s) = \begin{cases} 1 & \text{if} \quad s = 1 \\ 0 & \text{if} \quad s \geqslant 2 \end{cases}$$

then $\mathcal{S}_\varphi^{\text{qs}}(\xi) = \varphi$. Thus the inverse ψ of φ is given by the inverse of ξ in the group of characters :

$$\psi(x) = F_\psi.x = x + \sum_{s \geqslant 1, n_i \geqslant 1} \underbrace{(-1)^s}_{\xi^{*-1}(n_1, \ldots, n_s)} \mathbb{D}_{n_s} \ldots \mathbb{D}_{n_1}.x$$

$\mathrm{JC}_{k,d}$, quasishuffle and degree relations

Let $\varphi(x) = x + f(x) \in G$ as in $\mathrm{JC}_{k,d}$ ($d^\circ(f) = d$ and $J_\varphi^k = 0$), then

$$d^\circ(\mathbb{D}_{n_s} \ldots \mathbb{D}_{n_1}.x) = d^\circ(\mathbb{D}_{\boldsymbol{n}}.x) = 1 + \|\boldsymbol{n}\|(d-1)$$

where $\|\boldsymbol{n}\| = n_1 + \ldots + n_s$. If

$$l(n_1, \ldots, n_s) = l(\boldsymbol{n}) = s$$

then $\psi(x) = x + \sum_{\boldsymbol{n} \in \boldsymbol{N}/\{\emptyset\}} (-1)^{l(\boldsymbol{n})} \mathbb{D}_{\boldsymbol{n}}.x$ and the conjecture $\mathrm{JC}_{k,d}$ says that,

$$\exists n_0 \geqslant 1;\ \forall n \geqslant n_0,\ \psi_n = \sum_{\substack{\boldsymbol{n} \in \boldsymbol{N} \\ \|\boldsymbol{n}\| = n}} (-1)^{l(\boldsymbol{n})} \mathbb{D}_{\boldsymbol{n}}.x = 0$$

Note that ψ_n is the homogeneous component of ψ of degree $n(d-1)+1$.

First reduction

As

$$\mathbb{D}_n = \frac{1}{n!} \sum_{i_1, \ldots, i_n} f_{i_1}(x) \ldots f_{i_n}(x) \partial_{x_{i_1}} \ldots \partial_{x_{i_n}}$$

then $\mathbb{D}_n.x = 0$ if $n \geqslant 2$. More generally if $\boldsymbol{N}_d$ is

$$\{(n_1, \ldots n_s);\ \exists i_0 \geqslant 0; n_{i_0+1} > 1 + (d-1)(n_1 + \ldots + n_{i_0})\}$$

then, we have the degree relations

$$\forall \boldsymbol{n} \in \boldsymbol{N}_d, \quad \mathbb{D}_{\boldsymbol{n}}.x = 0$$

thus

$$\psi_n = \sum_{\substack{\boldsymbol{n} \in \boldsymbol{N}/\boldsymbol{N}_d \\ \|\boldsymbol{n}\| = n}} (-1)^{l(\boldsymbol{n})} \mathbb{D}_{\boldsymbol{n}}.x$$

Nilpotency and conjugacy of Vector Fields

Let $\varphi(x) = x + f(x)$ and $X^{\mathrm{lin}}(x) = \sum x_i \partial_{x_i}$. After the change of coordinates $y = \psi(x) = \varphi^{\circ -1}(x)$, $x_i = \varphi_i(y)$ and

$$\partial_{x_i} = \sum \frac{\partial \psi_j}{\partial x_i}(\varphi)\partial_{y_j} = \sum (J_\psi(\varphi))_{i,j}\partial_{y_j}$$

but

$$J_\psi(\varphi) = (J_\varphi)^{-1} = (I_\nu + J_f)^{-1} = \sum_{n \geqslant 0} (-1)^n J_f^n$$

The vector fields becomes

$$X(y) = \sum_{1 \leqslant i,j \leqslant \nu} \sum_{n \geqslant 0} (-1)^n (y_i + f_i(y))(J_f^n(y))_{i,j}\partial_{y_j}$$

and, if f is homogeneous of degree d,

$$\sum_{1 \leqslant i \leqslant \nu} y_i (J_f(y))_{i,j} = d\int_0^1 \sum_{1 \leqslant i \leqslant \nu} y_i (J_f(ty))_{i,j} dt = d f_j(y)$$

so

$$X(y) \;=\; X^{\mathrm{lin}}(y) + \sum_{n \geqslant 1} (-1)^n (d-1)\mathbb{B}_n$$

where

$$\mathbb{B}_n = \sum_{1 \leqslant i,j \leqslant \nu} f_i (J_f^{n-1})_{i,j}\partial_{y_j}$$

and the conjugacy equation reads, in given variables $x = (x_1, \ldots, x_\nu)$:

$$F_\varphi . X^{\mathrm{lin}} = X . F_\varphi \quad \text{or} \quad X^{\mathrm{lin}} . F_\psi = F_\psi . X$$

Here comes the shuffle Hopf algebra !

$\mathcal{SH}$ is the graded Hopf algebra whose linear basis is given by words

$$\boldsymbol{N} = \{\emptyset\} \cup \{\boldsymbol{n} = (n_1, \ldots, n_s),\ n_i \geqslant 1\}$$

Its product is the shuffle product :

$$\pi((n_1) \otimes (n_2, n_3)) = (n_1, n_2, n_3) + (n_2, n_1, n_3) + (n_2, n_3, n_1)$$

Its coproduct is given by the deconcatenation :

$$\Delta \boldsymbol{n} = \sum_{\boldsymbol{n}^1 . \boldsymbol{n}^2 = \boldsymbol{n}} \boldsymbol{n}^1 \otimes \boldsymbol{n}^2$$

Let $\varphi(x) = x + f(x)$ be in G (f homogeneous of degree d, and, for $n \geqslant 1$,

$$\mathbb{B}_n = \sum_{1 \leqslant i, j \leqslant \nu} f_i(x) (J_f^{n-1}(x))_{i,j} \partial_{x_j}$$

then

$$\begin{array}{rccc} \rho_\varphi^s : & \mathcal{SH}^* & \to & \mathbb{C}[x, \partial_x] \\ & \boldsymbol{n} = (n_1, \ldots, n_s) & \mapsto & \mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1} = \mathbb{B}_{\boldsymbol{n}} \end{array}$$

is an antialgebra morphism ($\rho_\varphi(\emptyset) = \mathbb{B}_\emptyset = \mathrm{Id}_{\mathbb{C}[[x]]}$) such that

$$\rho_\varphi^s(\boldsymbol{n})(AB) = \pi_{\mathbb{C}[[x]]} \circ ((\rho_\varphi \otimes \rho_\varphi) \circ \Delta(\boldsymbol{n}))(A \otimes B)$$

Now we are ready for more mould calculus.

Mould Calculus

Definition 4. *A symmetral mould (Ecalle) on* $\boldsymbol{N}$ *with values in* $\mathbb{C}$, $M^\bullet = \{M^{\boldsymbol{n}} \in \mathbb{C}, \boldsymbol{n} \in \boldsymbol{N}\}$ *is the collection of the values of a character* χ *of* $\mathcal{C}(\mathcal{SH}, \mathbb{C})$ *on its basis* ($M^{\boldsymbol{n}} = \chi(\boldsymbol{n})$)*. We have the product of moulds :*

$$(M_1^\bullet \times M_2^\bullet)^{\boldsymbol{n}} = (\chi_1 * \chi_2)(\boldsymbol{n}) = \pi_{\mathbb{C}} \circ (\chi_1 \otimes \chi_2) \circ \Delta(\boldsymbol{n})$$

Theorem 5. *Let* φ *in* G*, then the map*

$$\begin{array}{rccc} \mathcal{S}_\varphi^s & : \mathcal{C}(\mathcal{SH}, \mathbb{C}) & \rightarrow & G \\ & \chi & \mapsto & \displaystyle\sum_{\boldsymbol{n} \in \boldsymbol{N}} \chi(\boldsymbol{n}) \rho_\varphi^s(\boldsymbol{n}).x \end{array}$$

is a group morphism and the mould series :

$$\sum_{\boldsymbol{n} \in \boldsymbol{N}} \chi(\boldsymbol{n}) \rho_\varphi^s(\boldsymbol{n}) = \sum_{\boldsymbol{n} \in \boldsymbol{N}} M^{\boldsymbol{n}} \mathbb{B}_{\boldsymbol{n}}$$

is the substitution automorphism associated to $\mathcal{S}_\varphi^s(\chi)$

Moreover

$$\begin{aligned} \varphi(x) &= x + \sum \frac{(-1)^{n_1 + \ldots + n_s + s}}{n_1(n_1 + n_2) \ldots (n_1 + \ldots + n_s)} \mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1}.x \\ \psi(x) &= x + \sum \frac{(-1)^{n_1 + \ldots + n_s}}{n_s(n_{s-1} + n_s) \ldots (n_1 + \ldots + n_s)} \mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1}.x \end{aligned}$$

Proof

$F_\varphi . X^{\text{lin}} = X . F_\varphi$, and $[X^{\text{lin}}, \mathbb{B}_n] = n(d-1)\mathbb{B}_n$, thus,

$$[F_\varphi, X^{\text{lin}}] = (X - X^{\text{lin}})F_\varphi$$

but,

$$\begin{aligned}
[F_\varphi, X^{\text{lin}}] &= \sum_{\boldsymbol{n} \in \boldsymbol{N}/\{\emptyset\}} \sigma(\boldsymbol{n})[\mathbb{B}_{\boldsymbol{n}}, X^{\text{lin}}] \\
&= -\sum_{\boldsymbol{n} \in \boldsymbol{N}/\{\emptyset\}} \sigma(\boldsymbol{n})\|\boldsymbol{n}\|(d-1)\mathbb{B}_{\boldsymbol{n}}
\end{aligned}$$

$$\begin{aligned}
(X - X^{\text{lin}})F_\varphi &= \sum_{\substack{n_0 \geqslant 1 \\ \boldsymbol{n} \in \boldsymbol{N}}} (-1)^{n_0}(d-1)\mathbb{B}_{n_0} . \sigma(\boldsymbol{n})\mathbb{B}_{\boldsymbol{n}} \\
&= \sum_{\substack{n_0 \geqslant 1 \\ \boldsymbol{n} \in \boldsymbol{N}}} (-1)^{n_0}(d-1)\sigma(\boldsymbol{n})\mathbb{B}_{\boldsymbol{n}.n_0}
\end{aligned}$$

So, $\sigma(\emptyset) = 1$ and for $\boldsymbol{n} = (n_1, \ldots, n_s) \in \boldsymbol{N}/\{\emptyset\}$,

$$(n_1 + \ldots + n_s)\sigma(n_1, \ldots, n_s) = (-1)^{n_s+1}\sigma(n_1, \ldots, n_{s-1})$$

finally :

$$\sigma(n_1, \ldots, n_s) = \frac{(-1)^{\|\boldsymbol{n}\| + l(\boldsymbol{n})}}{n_1(n_1 + n_2)\ldots(n_1 + \ldots + n_s)}$$

and (antipode)

$$\sigma^{*-1}(n_1, \ldots, n_s) = \frac{(-1)^{\|\boldsymbol{n}\|}}{n_s(n_{s-1} + n_s)\ldots(n_1 + \ldots + n_s)}$$

$\mathrm{JC}_{k,d}$, shuffle and nilpotency relations

Let $\varphi(x) = x + f(x) \in G$ as in $\mathrm{JC}_{k,d}$ ($d^\circ(f) = d$ and $J_\varphi^k = 0$), then, if $n \geqslant k$, then

$$\mathbb{B}_n = \sum_{1 \leqslant i,j \leqslant \nu} f_i(x)(J_f^{n-1}(x))_{i,j}\partial_{x_j} = 0$$

and,

$$\mathbb{B}_n.\mathbb{B}_1.x = \mathbb{B}_{n+1}.x$$

If

$$\boldsymbol{N}_k = \{(n_1, ..., n_s)\,;\, \exists i_0\,;\, n_{i_0} \geqslant k\}$$

then

$$\forall \boldsymbol{n} \in \boldsymbol{N}_k, \quad \mathbb{B}_{\boldsymbol{n}} = \mathbb{B}_{\boldsymbol{n}}.x = 0$$

Finally, as

$$\begin{aligned} F_\varphi &= \mathrm{Id} + \sum_{n \geqslant 1} \mathbb{D}_n \\ &= \mathrm{Id} + \sum \underbrace{\frac{(-1)^{n_1+...+n_s+s}}{n_1(n_1+n_2)...(n_1+...+n_s)}}_{\sigma(n_1,...,n_s)} \mathbb{B}_{n_s}...\mathbb{B}_{n_1} \end{aligned}$$

We have, for $n \geqslant 1$,

$$\begin{aligned} \mathbb{D}_n &= \sum_{n_1+...+n_s=n} \frac{(-1)^{n_1+...+n_s+s}}{n_1(n_1+n_2)...(n_1+...+n_s)} \mathbb{B}_{n_s}...\mathbb{B}_{n_1} \\ \mathbb{B}_n &= \sum_{n_1+...+n_s=n} (-1)^{n_1+...+n_s+s}(n_1+...+n_s)\mathbb{D}_{n_s}...\mathbb{D}_{n_1} \end{aligned}$$

Summary

For $\varphi(x) = x + f(x) \in G$ as in $\mathrm{JC}_{k,d}$ ($d^\circ(f) = d$ and $J_\varphi^k = 0$), $\psi(x) = x + \sum \psi_n(x)$ ($d^\circ(\psi_n) = n(d-1)+1$).

1. $\psi_n = \sum_{\substack{\boldsymbol{n} \in \boldsymbol{N} \\ \|\boldsymbol{n}\| = n}} (-1)^{l(\boldsymbol{n})} \mathbb{D}_{\boldsymbol{n}}.x$,
2. $\forall \boldsymbol{n} \in \boldsymbol{N}_d, \quad \mathbb{D}_{\boldsymbol{n}}.x = 0$ (degree relations) ,
3. $\psi_n = \sum_{\substack{\boldsymbol{n} \in \boldsymbol{N} \\ \|\boldsymbol{n}\| = n}} \frac{(-1)^{n_1 + \ldots + n_s}}{n_s(n_{s-1} + n_s) \ldots (n_1 + \ldots + n_s)} \mathbb{B}_{\boldsymbol{n}}.x$
4. $\forall \boldsymbol{n} \in \boldsymbol{N}_k, \quad \mathbb{B}_{\boldsymbol{n}}.x = 0$ (Nilpotency) ,
5. A passage $\mathbb{D}_{\boldsymbol{n}} \leftrightarrow \mathbb{B}_{\boldsymbol{n}}$,
6. $\mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1}.\mathbb{B}_1.x = \mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1+1}.x$

Test on $\mathrm{JC}_{3,2}$

In this case, for $n \geqslant 6$, $\psi_n = 0$, and, using mupad-combinat or a sheet of paper, we get for $\|\boldsymbol{n}\| = 6$, $\mathbb{B}_{\boldsymbol{n}}.x = 0$ ... Except $\mathbb{B}_{2,2,2}.x$ and

$$\psi_6 = C \mathbb{B}_{2,2,2}.x \quad (C \neq 0)$$

We must look for hidden relations and will need trees.

Here comes the Connes–Kreimer Hopf algebra !

$\mathcal{CK}$ is the graded Hopf algebra whose linear basis is given by the set $\mathcal{F}$ of undecorated forests $F = T_1 \dots T_s$. Its product is obvious and its coproduct is given by admissible cuts or by

$$\Delta(B_+(F)) = B_+(F) \otimes \emptyset + (\mathrm{Id} \otimes B_+) \circ \Delta(F)$$

where, B_+ adds a common root to the trees in F.
Let $\mathbb{D}_\emptyset = \mathrm{Id}_{\mathbb{C}[[x]]}$,

$$\mathbb{D}_\bullet = \mathbb{D}_1 = \mathbb{B}_1 = \sum f_i(x) \partial_{x_i}$$

and (Ecalle's coarborification) :

- If $T = B_+(F)$,

$$\mathbb{D}_T = \sum (\mathbb{D}_F.(\mathbb{D}_\bullet.x_i))\partial_{x_i}$$

- If $F = T_1 \dots T_s$ $(s \geqslant 2)$ then

$$\mathbb{D}_F = \frac{1}{d_1! \dots d_t!} \sum ((\mathbb{D}_{T_1}.x_{i_1}) \dots (\mathbb{D}_{T_s}.x_{i_s}))\partial_{x_{i_1}} \dots \partial_{x_{i_s}}$$

  where $T_1 \dots T_s$ contains t distinct trees that appears d_j times.

then (F. Fauvet, F. M.), the map

$$\begin{array}{rcccc} \rho^{\mathrm{ck}}_\varphi & : & \mathcal{CK}^* & \to & \mathbb{C}[x, \partial_x] \\ & & F & \mapsto & \mathbb{D}_F \end{array}$$

is an algebra morphism.

Mould Calculus

Definition 6. *A separative arborescent mould (Ecalle) on $\mathcal{F}$ with values in $\mathbb{C}$, $M^\bullet = \{M^F \in \mathbb{C}, F \in \mathcal{F}\}$ is the collection of the values of a character χ of $\mathcal{C}(\mathcal{CK},\, \mathbb{C})$ on its basis ($M^F = \chi(F)$). We have the product of moulds :*

$$(M_1^\bullet \times M_2^\bullet)^F = (\chi_1 * \chi_2)(F) = \pi_{\mathbb{C}} \circ (\chi_1 \otimes \chi_2) \circ \Delta(F)$$

Theorem 7. *Let φ in G, then the map*

$$\begin{array}{rcl} \mathcal{S}_\varphi^{\mathrm{ck}} \; : \; \mathcal{C}(\mathcal{CK}, \mathbb{C}) & \to & G \\ \chi & \mapsto & \displaystyle\sum_{F \in \mathcal{F}} \chi(F) \rho_\varphi^{\mathrm{ck}}(F).x \end{array}$$

is a (anti)group morphism and the mould series :

$$\sum_{F \in \mathcal{F}} \chi(F) \rho_\varphi^{\mathrm{ck}}(F) = \sum_{F \in \mathcal{F}} M^F \mathbb{D}_F$$

is the substitution automorphism associated to $\mathcal{S}_\varphi^{\mathrm{ck}}(\chi)$.

As

$$\mathbb{D}_n = \mathbb{D}_{\underbrace{\bullet \ldots \bullet}_{n \text{ times}}}$$

φ is given by the character :

$$\tau(F) = \begin{cases} 1 & \text{if } F = \underbrace{\bullet \ldots \bullet}_{n \text{ times}} \\ 0 & \text{otherwise} \end{cases}$$

$\mathrm{JC}_{k,d}$ and the Connes–Kreimer H.A.

We have $\tau^{*-1}(F) = (-1)^{|F|}$ thus

$$\psi(x) = x + \sum_{|F| \geqslant 1} (-1)^{|F|} \mathbb{D}_F.x = x + \sum_{|T| \geqslant 1} (-1)^{|T|} \mathbb{D}_T.x$$

and

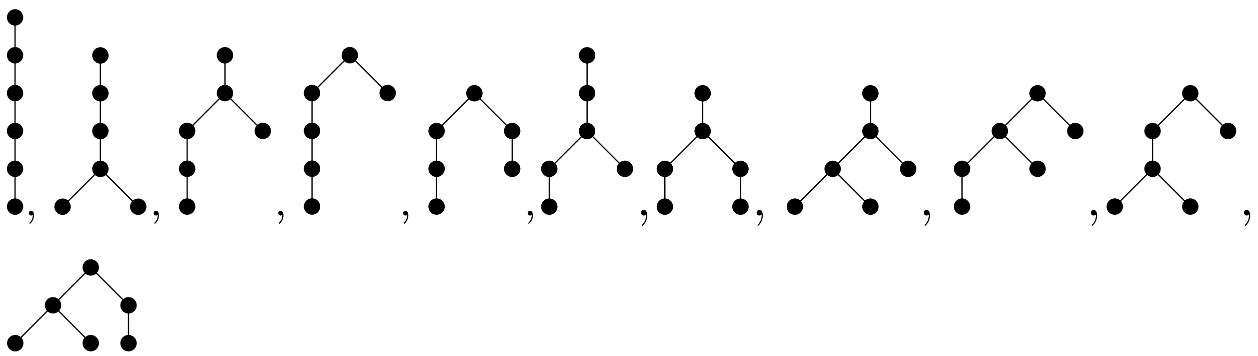
Degree relations : if a forest F has a vertex with at least $d+1$ outgoing edges, then $\mathbb{D}_F = 0$: In each term, one of the f_i, of degree d id differentiated at least $d+1$ times.

Nilpotency relations : in a forest $F = T_1 \dots T_s$, there exists a tree T_{i_0} such that, either $T_{i_0} = B_+^k(T')$ or, after one elementary cut C, $P^C(T_{i_0}) = B_+^k(T')$, then $\mathbb{D}_F = 0$.

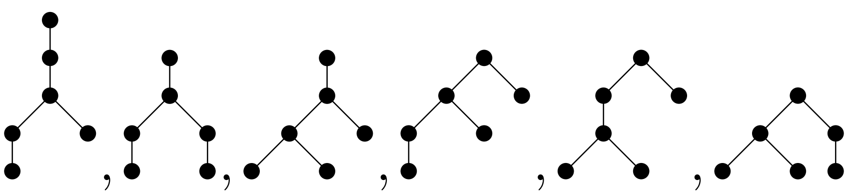
Now

$$(-1)^n \psi_n = \sum_{|T|=n} \mathbb{D}_T.x = \sum_{T \in \mathcal{T}_d^n} \mathbb{D}_T.x = \sum_{T \in \mathcal{T}_{d,k}^n} \mathbb{D}_T.x$$

For $\mathrm{JC}_{3,2}$ we have $\mathcal{T}_2^6$:



and $\mathcal{T}_{2,3}^6$:



The beginning of a strategy

Note that $\mathbb{B}_n = \mathbb{D}_{l_n}$ where l_n is the ladder $B_+^n(\emptyset)$ and the map

$$\begin{array}{rcccc} \alpha & : & \mathcal{SH}^* & \to & \mathcal{CK}^* \\ & & \boldsymbol{n} = (n_1, \ldots, n_s) & \mapsto & l_{n_1} \ldots l_{n_s} \end{array}$$

is a morphism of Hopf algebras and

$$\rho_\varphi^s(\boldsymbol{n}) = \mathbb{B}_{n_s} \ldots \mathbb{B}_{n_1} = \rho_\varphi^{\mathrm{ck}}(\alpha(n_s, \ldots, n_1)) = \rho_\varphi^{\mathrm{ck}}(\alpha(\tilde{\boldsymbol{n}}))$$

If

$$l_{n_1} \ldots l_{n_s} = \sum C_F^{\boldsymbol{n}} F$$

then

Theorem 8. *We have $C_\emptyset^\emptyset = 1$ and*

1. *If $F = T_1 \ldots T_s$ $(s \geqslant 2)$,*

$$C_F^{\boldsymbol{n}} = \sum_{\boldsymbol{n} = \mathrm{sh}(\boldsymbol{n}^1, \ldots, \boldsymbol{n}^s)} C_{T_1}^{\boldsymbol{n}^1} \ldots C_{T_s}^{\boldsymbol{n}^s}$$

2. *If $T = B_+(F)$,*

$$C_{B_+(F)}^{\boldsymbol{n}} = C_F^{B_-(\boldsymbol{n})}$$

where $B_-(\emptyset) = 0$ and

$$B_-(n_1, \ldots, n_s) = \begin{cases} (n_1, \ldots, n_{s-1}) & \text{if } n_s = 1 \\ (n_1, \ldots, n_s - 1) & \text{if } n_s \geqslant 2 \end{cases}$$

A strategy

We have

$$\psi_n = \sum_{\substack{\boldsymbol{n}\in\boldsymbol{N}/\boldsymbol{N}_k \\ \|\boldsymbol{n}\|=n}} \frac{(-1)^{n_1+\ldots+n_s}}{n_s(n_{s-1}+n_s)\ldots(n_1+\ldots+n_s)} \mathbb{B}_{\boldsymbol{n}}.x$$

1. Forget first that for $\boldsymbol{n}\in\boldsymbol{N}_k$, $\mathbb{B}_{\boldsymbol{n}}=0$.
2. Compute recursively

$$\mathbb{B}_{\boldsymbol{n}}.x = \sum C_F^{\tilde{\boldsymbol{n}}}\mathbb{D}_F.x = \sum C_T^{\tilde{\boldsymbol{n}}}\mathbb{D}_T.x = \sum_{T\in\mathcal{T}_{d,k}} C_T^{\tilde{\boldsymbol{n}}}\mathbb{D}_T.x$$

3a. Show that as elements of $\mathrm{Vect}\{\mathbb{D}_T.x, T\in\mathcal{T}_{d,k}\}$

$$\exists n_0\,;\forall n\geqslant n_0,\quad \psi_n\in\mathrm{Vect}\{\mathbb{B}_{\boldsymbol{n}}.x,\boldsymbol{n}\in\boldsymbol{N}_k\}$$

In fact one has to check for $n=n_0,\ldots,2(n_0-1)$.

3b. (Stronger)

$\exists n_0\,;\forall\boldsymbol{m}\in\boldsymbol{N}/\boldsymbol{N}_k;\|\boldsymbol{m}\|\geqslant n_0, \mathbb{B}_{\boldsymbol{m}}.x\in\mathrm{Vect}\{\mathbb{B}_{\boldsymbol{n}}.x,\boldsymbol{n}\in\boldsymbol{N}_k\}$

In fact one has to check for $\|\boldsymbol{n}\|=n_0,\ldots,n_0+k-1$.

3c. The same but just with the $\mathbb{D}_T.x$ ...

Test on $\mathrm{JC}_{3,2}$

This work !!

Test on $\mathrm{JC}_{4,2}$ and $\mathrm{JC}_{3,3}$

Computations up to $n=14$ but this is not sufficient.

Conclusion

All this requires

1. A better computer to test for greater values of n.
2. A better programmer, then, perhaps, no better computer is needed.
3. A better thinker : There is a lot of algebraic structures here and there maybe a nice combinatorial way to exhausts all the relations that lead to the result. But at one stage, one has to do some linear algebra, and, as the Gaussian Elimination is quite arbitrary, one can only guess the combinatorial structure of these relations.