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GRT EQUATIONS OF THE EARTH'S ROTATION

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In spite of many attempts we still have no reliable GRT equations of the Earth's rotation to be used in practice (see, for example, Klioner, 1996 and references therein). Meanwhile the Newtonian theory of the rigid Earth rotation is presently advanced so far that to take into account the GRT effects is of practical necessity (Bretagnon et al., 1997).

The most modern theoretical results concerning the problem of rotation in the GRT framework have been obtained by Damour, Soffel and Xu (1993). But for practical application one should have a specific system of ordinary differential equations describing the rotation of the Earth in some definite coordinate system. The first question is whether one has to forget all results obtained in this problem in the framework of Infeld and Fock schools and to apply inclusively the DSX approach? It seems that one cannot treat all former results as pure formal being of no interest nowadays. First of all, the GRT approach developed by Infeld is the most economical one as compared with all other approaches. Indeed, the post-Newtonian equations of translatory and rotational motion of celestial bodies may be obtained from the action principle resulting both in the field equations and the equations of motion. It was shown by Infeld for the delta-function stress-tensor (Infeld and Plebański, 1960) and extended by Brumberg (1972) for the liquid body stress-tensor. In this approach one has to deal only with components $h_{00}^{(2)}$, $h_{ij}^{(2)}$ and $h_{0i}^{(3)}$ whereas all other existing techniques demand $h_{00}^{(4)}$ as well (numbers in parentheses indicate the order of smallness with respect to q^2/c^2 or U/c^2, q and U being the characteristic velocity of the celestial bodies and Newtonian gravitational potential, respectively). The first three components may be expressed only in terms of Newtonian potential and Newtonian vector-potential. This means that in post-Newtonian celestial mechanics one may deal without BD moments at all (needless to say that BD moments are useful to present the GRT equations of motion in more compact form but they are not needed to derive these equations). This is evident for BRS treatment (one may remember that the post-Newtonian BRS equations of light propagation do not demand $h_{00}^{(4)}$ as well). As for GRS treatment is concerned one may derive GRS equations also just from the action principle written in GRS or by transforming BRS equations using BRS→GRS post-Newtonian transformation. In both cases the GRS component $\hat{h}_{00}^{(4)}$ is not needed. One of the consequences of this fact is that the present IAU (1991) resolutions on reference systems and time scales should be specified by adding only $h_{0i}^{(3)}$ (and not $h_{00}^{(4)}$) to the explicitly given $h_{00}^{(2)}$ and $h_{ij}^{(2)}$.

There is no doubt that GRT equations of the Earth's rotation have the simplest form in DGRS, dynamically nonrotating GRS. It seems desirable to derive the corresponding equations in two independent ways, i.e. directly in DGRS and in BRS accompanied by BRS→GRS transformation. In applying latter approach one should take into account the transformations relating BRS and GRT quantities such as angular velocity, rigid body velocity distribution, quadrupole moments, mutual distances and so on (Brumberg, 1995a,b; 1997). In other words one should derive the GRT equations of the Earth's rotation just in the ways as were applied by Brumberg and Kopeikin (1989) to derive Earth satellite equations of motion.

In dealing with the BRS approach one starts with the BRS equations of the Earth rotation (see Brumberg, 1968, 1972 with improvements in Pushkarev and Abdil'din, 1976)

$$\frac{dS_E^k}{dt} = Q_E^k \tag{1}$$

with

$$S_E^k = I_E^{ii}\omega_E^k - I_E^{ik}\omega_E^i + c^{-2}R_E^k \tag{2}$$

and right-hand members Q_E^k beginning with the leading Newtonian quadrupole terms

$$Q_{E(N)}^k = 3\varepsilon_{ijk}I_E^{is}\sum_{A\neq E}\frac{GM_A}{r_{EA}^5}r_{EA}^s r_{EA}^j + \dots\,. \tag{3}$$

Here and everywhere below the Einstein summation rule from 1 to 3 is applied to each repeated latin index. Capital latin letters relate to the Earth (E) and disturbing bodies (A) such as Sun and Moon for the first instance. The use of the Levi-Civita fully antisymmetric symbol

$$\varepsilon_{ijk} = \tfrac{1}{2}(i-j)(j-k)(k-i)$$

simplifies algebra manipulation due to the identity

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}\,,$$

δ_{ij} being the Kronecker symbol. In terms of ε_{ijk} the component i of the vector product $\mathbf{c} = \mathbf{a}\times\mathbf{b}$ is $c^i = \varepsilon_{ijk}a^j b^k$.

The BRS quantities occurring in the Newtonian part of (2) may be expressed in terms of their GRS counterparts as follows (Brumberg, 1995a,b):

$$\begin{aligned} I_E^{ik}(t^*) = {} & \left[1 - 2c^{-2}\bar{U}(t,\mathbf{x}_E)\right]\hat{I}_E^{ik}(u) - \frac{1}{2}c^{-2}v_E^m(v_E^i\hat{I}_E^{km} + v_E^k\hat{I}_E^{im}) - \\ & - c^{-2}(\varepsilon_{imn}\hat{I}_E^{km} + \varepsilon_{kmn}\hat{I}_E^{im})F^n \end{aligned} \tag{4}$$

and

$$\omega_E^i(t^*) = \hat{\omega}_E^i(u) + c^{-2}\Big[-\big(\tfrac{1}{2}v_E^2 + \bar{U}_E(\mathbf{x}_E)\big)\hat{\omega}_E^i + \varepsilon_{ijk}F^j\hat{\omega}_E^k + \dot{F}^i\Big]\,, \tag{5}$$

t^* being a moment of TCB$=t$ related to TCG$=u$ by the equation

$$u = t^* - c^{-2}A\,, \qquad \dot{A} = \tfrac{1}{2}v_E^2 + \bar{U}_E(\mathbf{x}_E) \tag{6}$$

and F^i representing the geodesic rotation vector. Substituting (4) and (5) into (2) one gets

$$S_E^k = \hat{I}_E^{ii}\hat{\omega}_E^k - \hat{I}_E^{ik}\hat{\omega}_E^i + c^{-2}\hat{R}_E^k\,. \tag{7}$$

Equations (1) may be regarded as related to t^*. Changing the independent argument in accordance with (6) one obtains the DGRS equations of the Earth's rotation in the form

$$\frac{dS_E^k}{du} = \hat{Q}_E^k \tag{8}$$

with

$$\hat{Q}^k_E = Q^k_E(1 + c^{-2}\dot{A}). \tag{9}$$

It remains to convert these equations into GRS$^+$ system rotating with the Earth. This system has the same time argument u but its spatial coordinates W^i are resulted from the three-dimensional rotation

$$W^i = \hat{P}_{ik} w^k \,. \tag{10}$$

This transformation involves

$$T^i_E = \hat{P}_{ik} S^k_E \,, \qquad M^i_E = \hat{P}_{ik} \hat{Q}^k_E \,, \qquad \mathcal{F}^i = \hat{P}_{ik} F^k \,, \qquad \hat{\Omega}^i_E = \hat{P}_{ik} \hat{\omega}^k_E \tag{11}$$

and constant inertia coefficients $\hat{J}^{ij}_E$

$$\hat{J}^{ij}_E = \hat{P}_{ir} \hat{P}_{js} \hat{I}^{rs}_E \,. \tag{12}$$

Identifying the statial axes of GRS$^+$ with the principal axes of inertia of the Earth one gets the diagonal matrix of inertia coefficients with

$$\hat{J}^{11}_E = \tfrac{1}{2}(B + C - A), \qquad \hat{J}^{22}_E = \tfrac{1}{2}(C + A - B), \qquad \hat{J}^{33}_E = \tfrac{1}{2}(A + B - C), \tag{13}$$

A, B, C being the principal moments of inertia. Functions T^i_E are determined by relations

$$T^i_E = \hat{J}^{ss}_E \hat{\Omega}^i_E - \hat{J}^{is}_E \hat{\Omega}^s_E + c^{-2} \mathcal{T}^i_E \,, \qquad \mathcal{T}^i_E = \hat{P}_{ik} \hat{R}^k_E \tag{14}$$

and satisfy the equations

$$\frac{dT^i_E}{du} + \varepsilon_{ijk} \hat{\Omega}^j_E T^k_E = M^i_E \,. \tag{15}$$

Therefore, GRT equations of the Earth's rotation replacing the classical Euler equations are of the form

$$\hat{J}^{ss}_E \frac{d\hat{\Omega}^i_E}{du} - \hat{J}^{is}_E \frac{d\hat{\Omega}^s_E}{du} - \varepsilon_{ijk} \hat{J}^{ks}_E \hat{\Omega}^s_E \hat{\Omega}^j_E = N^i_E \tag{16}$$

with

$$N^i_E = M^i_E - c^{-2} \left(\dot{\mathcal{T}}^i_E + \varepsilon_{ijk} \hat{\Omega}^j_E \mathcal{T}^k_E \right) . \tag{17}$$

Remembering (Brumberg, 1995a) that

$$\dot{\hat{P}}_{ik} = \varepsilon_{imj} \hat{\Omega}^j \hat{P}_{mk} \tag{18}$$

one finds

$$\dot{\mathcal{T}}^i_E + \varepsilon_{ijk} \hat{\Omega}^j_E \mathcal{T}^k_E = \hat{P}_{ik} \dot{\hat{R}}^k_E \tag{19}$$

and finally

$$N^i_E = \hat{P}_{ik} \big(\hat{Q}^k_E - c^{-2} \dot{\hat{R}}^k_E \big). \tag{20}$$

The value of S^k_E given in (Brumberg, 1968, 1972; Pushkarev and Abdil'din, 1976) should be corrected for the difference of the BRS Earth velocity distribution from the rigid body distribution. Indeed, the DGRS rigid body Earth velocity distribution

$$\hat{v}^i = \varepsilon_{ijk}\hat{\omega}^j_E w^k \tag{21}$$

involves the BRS distribution (Brumberg, 1995a)

$$v^i - v^i_E = \varepsilon_{ijk}\omega^j_E(x^k - x^k_E) + c^{-2} f^i_E \tag{22}$$

with

$$f^i_E = \varepsilon_{ijk}\Phi^j_E r^k_E - \left[\tfrac{1}{2}\left(a^i_E v^k_E + a^k_E v^i_E\right) + \tfrac{1}{4}\left(\dot{a}^i_E r^k_E + \dot{a}^k_E r^i_E\right) + \dot{D}^{ik}\right] r^k_E - \\ - \tfrac{1}{2} v^i_E \varepsilon_{kjm} v^k_E \omega^j_E r^m_E + \tfrac{1}{2} v^m_E r^m_E \varepsilon_{ijk}\omega^j_E v^k_E + \tfrac{1}{2} r^m_E r^m_E \varepsilon_{ijk}\omega^j_E a^k_E \tag{23}$$

and

$$\Phi^j_E = \left(a^j_E r^s_E - a^s_E r^j_E\right)\omega^s_E - \left(\varepsilon_{mns} v^m_E \omega^n_E r^s_E + 2 a^m_E r^m_E\right)\omega^j_E + \tfrac{3}{4}\varepsilon_{mnj}\dot{a}^n_E r^m_E \,. \tag{24}$$

Therefore, the contribution to R^k_E due to f^i_E ignored in (Brumberg, 1968, 1972; Pushkarev and Abdil'din, 1976) is as follows:

$$\delta_f R^k_E = \varepsilon_{ijk}\int_{(E)} \rho r^i_E f^j_E \, d^3x = \\ = \tfrac{1}{2} I^{is}_E \Big[-\varepsilon_{kij}(a^j_E v^s_E + a^s_E v^j_E + \varepsilon_{smn} v^j_E v^m_E \omega^n_E) + v^s_E(v^i_E \omega^k_E - v^k_E \omega^i_E)\Big]. \tag{25}$$

It remains to take into account the correction terms due to pressure p_{ij} inside the Earth. Proceeding as in (Pushkarev and Abdil'din, 1976) let us denote

$$p^{(E)}_{<ij>} = \int_{(E)} p_{ij} d^3x \,,$$

$$p^{(E)}_{<ijm>} = \int_{(E)} (x^m - x^m_E) p_{ij} d^3x \,, \tag{26}$$

$$p^{(E)}_{<ijmn>} = \int_{(E)} (x^m - x^m_E)(x^n - x^n_E) p_{ij} d^3x \,.$$

From the general expressions for S^k_E and Q^k_E given in (Brumberg, 1968, 1972) it is seen that the corrections due to pressure are

$$c^2 \delta_p S^k_E = \varepsilon_{kij}\left(v^m_E p^{(E)}_{<imj>} + \varepsilon_{mns}\omega^n_E p^{(E)}_{<imjs>}\right) \tag{27}$$

and

$$c^2 \delta_p Q^k_E = \varepsilon_{kij} \sum_{A \neq E} \frac{GM_A}{r^3_{EA}} \left(p^{(E)}_{<ssi>} r^j_{EA} + \frac{3}{r^2_{EA}} p^{(E)}_{<ssjn>} r^i_{EA} r^n_{EA}\right) + \\ + \varepsilon_{kij}\left(v^m_E v^i_E p^{(E)}_{<jm>} + \varepsilon_{mns}\omega^n_E v^i_E p^{(E)}_{<jms>}\right). \tag{28}$$

Combining these corrections with the value of S_E^k and Q_E^k found in (Brumberg, 1968, 1972; Pushkarev and Abdil'din, 1976) one gets for the post-Newtonian quadrupole approximation (omitting all terms dependent only on the internal structure of the Earth to affect its mass and quadrupole moments and treating the Sun and the Moon as the point masses)

$$\begin{aligned} R_E^k &= \delta_f R_E^k + c^2\delta_p S_E^k + \tfrac{1}{2}v_E^2(I_E^{ii}\omega_E^k - I_E^{ik}\omega_E^i) + \varepsilon_{smn}v_E^m\omega_E^n\varepsilon_{ijk}I_E^{is}v_E^j + \\ &+ \sum_{A\neq E}\frac{GM_A}{r_{EA}}\Big[3(I_E^{ii}\omega_E^k - I_E^{ik}\omega_E^i) - \frac{1}{r_{EA}^2}\varepsilon_{kij}(3v_E^j - 4v_A^j)I_E^{is}r_{EA}^s - \\ &- \frac{1}{2r_{EA}^2}\varepsilon_{kij}I_E^{is}(v_A^s r_{EA}^j + v_A^j r_{EA}^s) + \frac{3}{2r_{EA}^4}\varepsilon_{kij}I_E^{is}r_{EA}^j r_{EA}^s r_{EA}^n v_A^n\Big] \end{aligned} \quad (29)$$

and

$$\begin{aligned} Q_E^k &= Q_{E(N)}^k + \delta_p Q_E^k + c^{-2}\varepsilon_{lmn}v_E^m\omega_E^n v_E^s\big(\omega_E^s I_E^{lk} - \omega_E^k I_E^{ls}\big) + \\ &+ c^{-2}\sum_{A\neq E}\frac{GM_A}{r_{EA}^3}\Big\{(4v_A^m - 3v_E^m)(\omega_E^m I_E^{sk} - \omega_E^k I_E^{sm} + \varepsilon_{jmn}\varepsilon_{kis}\omega_E^n I_E^{ij})r_{EA}^s + \\ &+ \tfrac{1}{2}\varepsilon_{kij}\dot{I}_E^{is}\Big(-v_A^s r_{EA}^j - v_A^j r_{EA}^s + \frac{3}{r_{EA}^2}r_{EA}^s r_{EA}^j r_{EA}^n v_A^n\Big)\Big\} + \\ &+ c^{-2}\sum_{A\neq E}GM_A\Big\{\frac{3}{r_{EA}^5}\varepsilon_{kij}I_E^{is}r_{EA}^s r_{EA}^j\big[\tfrac{1}{2}(3v_E^2 + 3v_A^2 - 8v_E^n v_A^n) - \bar{U}_A(\mathbf{x}_A)\big] + \\ &+ \tfrac{1}{2}\varepsilon_{kij}v_A^m v_E^s I_E^{in}\frac{\partial^4 r_{EA}}{\partial x_E^j\partial x_E^s\partial x_E^m\partial x_E^n}\Big\} + \tfrac{1}{2}c^{-2}\varepsilon_{kij}I_E^{js}\frac{\partial^2}{\partial x_E^i\partial x_E^s}\big(\bar{U}_E(\mathbf{x}_E)\big)^2 . \end{aligned} \quad (30)$$

Before proceeding further let us remind the Newtonian relation

$$\dot{I}_E^{ij} = (\varepsilon_{kin}I_E^{kj} + \varepsilon_{kjn}I_E^{ki})\omega_E^n \quad (31)$$

(applied in fact to DGRS quantities) and the expression for the geodesic rotation

$$\dot{F}^k = \tfrac{1}{2}\varepsilon_{kij}\sum_{A\neq E}\frac{GM_A}{r_{EA}^3}\Big(3v_E^j - 4v_A^j\Big)r_{EA}^i \quad (32)$$

or else

$$\varepsilon_{knm}\dot{F}^m = \tfrac{1}{2}\sum_{A\neq E}\frac{GM_A}{r_{EA}^3}\Big[(4v_A^k - 3v_E^k)r_{EA}^n - (4v_A^n - 3v_E^n)r_{EA}^k\Big]. \quad (33)$$

Then one should estimate the analytical order of smallness of the corresponding terms. If M is a characteristic mass of a body, R is its characteristic size, q is its characteristic velocity and D is a characteristic distance between bodies, then evidently

$$GM_A \sim q^2 D, \qquad \omega_E \sim q/R, \qquad \dot{\omega}_E \sim q^2/D^2\,, \qquad I_E \sim MR^2\,, \qquad \dot{I}_E \sim MqR\,. \quad (34)$$

R^k_E and $\mathcal{T}^i_E$ should be determined up to the order Mq^3R^3/D^2 inclusively in correspondence to the order $c^{-2}Mq^4R^2/D^2$ in Q^k_E and N^i_E. The pressure integrals may be taken by a technique due to Fock (1955). On the one hand one has the Newtonian internal equations of motion

$$\rho\ddot{x}^i_E - \rho\frac{\partial\bar{U}_E(\mathbf{x}_E)}{\partial x^i_E} - \rho\frac{\partial u_E}{\partial x^i} + \\ + \rho\Big(\varepsilon_{ins}\dot{\omega}^n - \omega^2_E\delta_{is} + \omega^i_E\omega^s_E - \frac{\partial^2\bar{U}_E(\mathbf{x}_E)}{\partial x^i_E\partial x^s_E}\Big)(x^s - x^s_E) = \frac{\partial p_{ir}}{\partial x^r}\,. \qquad (35)$$

The first two terms mutually cancel within adopted accuracy. The third term affecting the internal structure terms may be removed in the process of integration. On the other hand there exist identities

$$2p_{ik} + x^i\frac{\partial p_{kr}}{\partial x^r} + x^k\frac{\partial p_{ir}}{\partial x^r} = \frac{\partial}{\partial x^r}\Big(x^i p_{kr} + x^k p_{ir}\Big), \qquad (36)$$

$$2x^j p_{ik} + x^j\Big(x^i\frac{\partial p_{kr}}{\partial x^r} + x^k\frac{\partial p_{ir}}{\partial x^r}\Big) - x^i x^k\frac{\partial p_{jr}}{\partial x^r} = \frac{\partial}{\partial x^r}\Big(x^j x^i p_{kr} + x^j x^k p_{is} - x^i x^k p_{js}\Big)\,, \qquad (37)$$

and so on. In actually using these identities x^i should be relaced by $x^i - x^i_E$. The right-hand members of (36), (37), etc., give no effect in integration. Moreover, within the quadrupole approximation only $< p_{ik} >$ has an actual contribution. There results

$$p_{<ik>} = I^{ik}_E\omega^2_E - \tfrac{1}{2}\omega^s_E\big(I^{is}_E\omega^k_E + I^{ks}_E\omega^i_E\big) - \tfrac{1}{2}\big(I^{is}_E\varepsilon_{kns} + I^{ks}_E\varepsilon_{ins}\big)\dot{\omega}^n_E + \\ + \sum_{A\neq E}\frac{GM_A}{r^3_{EA}}\Big[-I^{ik}_E + \frac{3}{2r^2_{EA}}r^s_{EA}\big(I^{is}_E r^k_{EA} + I^{ks}_E r^i_{EA}\big)\Big]. \qquad (38)$$

Performing transformation (4) and (5) into (2) one gets the correction

$$c^2\delta S^k_E = \big[\tfrac{1}{2}v^2_E + 3\bar{U}_E(\mathbf{x}_E)\big]\big(\hat{I}^{ks}_E\hat{\omega}^s_E - \hat{I}^{ss}_E\hat{\omega}^k_E\big) - v^r_E v^s_E\hat{I}^{rs}_E\hat{\omega}^k_E + \\ + \tfrac{1}{2}v^m_E\hat{\omega}^n_E\big(v^n_E\hat{I}^{km}_E + v^k_E\hat{I}^{mn}_E\big) + \dot{F}^k\hat{I}^{ss}_E - \dot{F}^s\hat{I}^{ks}_E + \\ + \big(\varepsilon_{kmn}\hat{I}^{ss}_E - \varepsilon_{kms}\hat{I}^{ns}_E\big)\hat{\omega}^n_E F^m. \qquad (39)$$

Combining this expression with the substitution of (25) into (29) one has

$$\hat{R}^k_E = \varepsilon_{kmn}\Big(\hat{\omega}^n_E\hat{I}^{ss}_E - \hat{\omega}^s_E\hat{I}^{ns}_E\Big)F^m + \\ + \tfrac{1}{2}\Big(\varepsilon_{ink}\varepsilon_{smj}\hat{I}^{is}_E\hat{\omega}^j_E + \hat{I}^{km}_E\hat{\omega}^n_E - \hat{I}^{mn}_E\hat{\omega}^k_E\Big)v^m_E v^n_E + \\ + \sum_{A\neq E}\frac{GM_A}{r^3_{EA}}\Big[\big(\tfrac{1}{2}\varepsilon_{kij}\hat{I}^{ss}_E - \tfrac{1}{2}\varepsilon_{sij}\hat{I}^{ks}_E - \varepsilon_{ksj}\hat{I}^{is}_E\big)(3v^j_E - 4v^j_A)r^i_{EA} + \\ + \tfrac{1}{2}(\varepsilon_{kij}\hat{I}^{is}_E + \varepsilon_{kis}\hat{I}^{ij}_E)(v^j_E - v^j_A)r^s_{EA} + \frac{3}{2r^2_{EA}}\varepsilon_{kij}\hat{I}^{is}_E r^j_{EA}r^s_{EA}r^n_{EA}v^n_A\Big]. \qquad (40)$$

Taking into account estimations (34) one gets the derivative of (40) within the accuracy Mq^4R^2/D^2 as follows:

$$
\begin{aligned}
\dot{R}^k_E = &\varepsilon_{kmn}\left(\hat\omega^n_E\dot{\hat I}^{ss}_E - \hat\omega^s_E\dot{\hat I}^{ns}_E + \dot{\hat\omega}^n_E\hat I^{ss}_E - \dot{\hat\omega}^s_E\hat I^{ns}_E\right)F^m +\\
&+\varepsilon_{kmn}\left(\hat\omega^n_E\hat I^{ss}_E - \hat\omega^s_E\hat I^{ns}_E\right)\dot F^m +\\
&+\tfrac{1}{2}\left(\varepsilon_{ink}\varepsilon_{smj}\dot{\hat I}^{is}_E\hat\omega^j_E + \dot{\hat I}^{km}_E\hat\omega^n_E - \dot{\hat I}^{mn}_E\hat\omega^k_E\right)v^m_E v^n_E +\\
&+\tfrac{1}{2}\left(\varepsilon_{ink}\varepsilon_{smj}\hat I^{is}_E\dot{\hat\omega}^j_E + \hat I^{km}_E\dot{\hat\omega}^n_E - \hat I^{mn}_E\dot{\hat\omega}^k_E\right)v^m_E v^n_E +\\
&+\sum_{A\neq E}\frac{GM_A}{r^3_{EA}}\Big[\big(\tfrac{1}{2}\varepsilon_{kij}\dot{\hat I}^{ss}_E - \tfrac{1}{2}\varepsilon_{sij}\dot{\hat I}^{ks}_E - \varepsilon_{ksj}\dot{\hat I}^{is}_E\big)(3v^j_E - 4v^j_A)r^i_{EA} +\\
&+\tfrac{1}{2}(\varepsilon_{kij}\dot{\hat I}^{is}_E + \varepsilon_{kis}\dot{\hat I}^{ij}_E)(v^j_E - v^j_A)r^s_{EA} + \frac{3}{2r^2_{EA}}\varepsilon_{kij}\dot{\hat I}^{is}_E r^j_{EA}r^s_{EA}r^n_{EA}v^n_A -\\
&-\tfrac{1}{2}\left(\varepsilon_{ink}\varepsilon_{smj}\hat I^{is}_E\hat\omega^j_E + \hat I^{km}_E\hat\omega^n_E - \hat I^{mn}_E\hat\omega^k_E\right)\left(v^m_E r^n_{EA} + v^n_E r^m_{EA}\right)\Big]. \qquad (41)
\end{aligned}
$$

Using (31) one gets after some algebra manipulation

$$
\begin{aligned}
\dot{R}^k_E = &\varepsilon_{kmn}\left(\hat\omega^n_E\hat I^{ss}_E - \hat\omega^s_E\hat I^{ns}_E\right)\dot F^m +\\
&+\left(\hat\omega^m_E\hat I^{ks}_E - \hat\omega^k_E\hat I^{ms}_E\right)\hat\omega^s_E F^m + \varepsilon_{kmn}\left(\dot{\hat\omega}^n_E\hat I^{ss}_E - \dot{\hat\omega}^s_E\hat I^{ns}_E\right)F^m +\\
&+\tfrac{1}{2}\hat I^{rs}_E\sum_{A\neq E}\frac{GM_A}{r^3_{EA}}\Big[\delta_{kr}(4v^n_A - 3v^n_E)r^s_{EA}\hat\omega^n_E + \delta_{kr}(4v^s_A - 3v^s_E)r^n_{EA}\hat\omega^n_E -\\
&-2(4v^r_A - 3v^r_E)r^s_{EA}\hat\omega^k_E + 2\varepsilon_{rin}\varepsilon_{sjk}(4v^j_A - 3v^j_E)r^i_{EA}\hat\omega^n_E + \varepsilon_{knr}\varepsilon_{sij}(4v^j_A - 3v^j_E)r^i_{EA}\hat\omega^n_E +\\
&+2v^r_A r^s_{EA}\hat\omega^k_E - \delta_{kr}(v^s_A r^n_{EA} + v^n_A r^s_{EA})\hat\omega^n_E - \varepsilon_{rnk}\varepsilon_{smj}(v^m_A r^n_{EA} + v^n_A r^m_{EA})\hat\omega^j_E +\\
&+\frac{3}{r^2_{EA}}\left(\varepsilon_{sjk}\varepsilon_{rin}\hat\omega^n_E r^i_{EA}r^j_{EA} + \delta_{kr}\hat\omega^j_E r^j_{EA}r^s_{EA} - \hat\omega^k_E r^k_{EA}r^s_{EA}\right)r^m_{EA}v^m_A\Big] +\\
&+\tfrac{1}{2}v^m_E v^n_E\hat\omega^j_E\Big[\big(\varepsilon_{rkj}\hat I^{rm}_E + 2\varepsilon_{rmj}\hat I^{rk}_E - \varepsilon_{kmi}\hat I^{ij}_E\big)\hat\omega^n_E + \big(\varepsilon_{kms}\hat\omega^j + 3\varepsilon_{jms}\hat\omega^k_E\big)\hat I^{ns}_E\Big] +\\
&+\tfrac{1}{2}\left(\varepsilon_{ink}\varepsilon_{smj}\hat I^{is}_E\dot{\hat\omega}^j_E + \hat I^{km}_E\dot{\hat\omega}^n_E - \hat I^{mn}_E\dot{\hat\omega}^k_E\right)v^m_E v^n_E\,. \qquad (42)
\end{aligned}
$$

To calculate M^i_E one should start with the transformation of the Newtonian part (3) replacing not only the BRS inertia moments by (4) but also the BRS distances in accordance with the relations (Brumberg, 1995b)

$$
\begin{aligned}
r^i_{AE}(t^*) = &w^i_A(u) + c^{-2}\Big[v^m_E r^m_{AE}(\tfrac{1}{2}v^i_E - v^i_A) + \varepsilon_{ijk}F^j r^k_{AE} - \bar U_E(\mathbf{x}_E)r^i_{AE} -\\
&- a^m_E r^m_{AE}r^i_{AE} + \tfrac{1}{2}r^2_{AE}a^i_E\Big] \qquad (43)
\end{aligned}
$$

and

$$
r_{AE}(t^*) = w_A\left\{1 + c^{-2}\Big[\frac{1}{w^2_A}v^m_E w^m_A(\tfrac{1}{2}v^n_E - v^n_A)w^n_A - \bar U_E(\mathbf{x}_E) - \tfrac{1}{2}a^m_E w^m_A\Big]\right\} \qquad (44)
$$

with Newtonian values in relativistic parts

$$w_A = r_{EA}\,, \qquad w_A^i = -r_{EA}^i = x_A^i - x_E^i\,, \qquad a_E^i = \sum_{A\neq E} \frac{GM_A}{w_A^3} w_A^i \tag{45}$$

and $W_A^i = \hat{P}_{ik} w_A^k$ in accordance with (10). Transformations (4) and (43) involve the corrections $\delta_I Q_E^k$ and $\delta_D Q_E^k$, respectively,

$$Q_{E(N)}^k = \hat{Q}_{E(N)}^k + \delta_I Q_E^k + \delta_D Q_E^k \tag{46}$$

with

$$\hat{Q}_{E(N)}^k = 3\varepsilon_{ijk} \hat{I}_E^{is} \sum_{A\neq E} \frac{GM_A}{w_A^5} w_A^s w_A^j + \ldots\,, \tag{47}$$

$$\delta_I Q_E^k = -3c^{-2} \varepsilon_{ijk} \sum_{A\neq E} \frac{GM_A}{w_A^5} w_A^j w_A^s \Big[2\bar{U}_E(\mathbf{x}_E) I_E^{is} + \tfrac{1}{2} v_E^m \big(v_E^i I_E^{sm} + v_E^s I_E^{im} \big) + \\ + F^n \big(\varepsilon_{imn} I_E^{sm} + \varepsilon_{smn} I_E^{im} \big) \Big] \tag{48}$$

and

$$\delta_D Q_E^k = 3c^{-2} \varepsilon_{ijk} I_E^{is} \sum_{A\neq E} \frac{GM_A}{w_A^5} \Big\{ w_A^j w_A^s \Big[3\bar{U}_E(\mathbf{x}_E) + \tfrac{1}{2} w_A^m a_E^m + \\ + \frac{5}{w_A^2} v_E^m w_A^m \big(v_A^n - \tfrac{1}{2} v_E^n \big) \Big] + \tfrac{1}{2} w_A^2 (w_A^j a_E^s + w_A^s a_E^j) - \\ - v_E^m w_A^m (w_A^j v_A^s + w_A^s v_A^j) + \tfrac{1}{2} v_E^m w_A^m (w_A^j v_E^s + w_A^s v_E^j) + \\ + F^m w_A^n (\varepsilon_{jmn} w_A^s + \varepsilon_{smn} w_A^j) \Big\}. \tag{49}$$

One should add to this one more correction due to transformation (9) of the time-argument

$$\delta_t Q_E^k = 3c^{-2} \varepsilon_{ijk} I_E^{is} \Big(\tfrac{1}{2} v_E^2 + \bar{U}_E(\mathbf{x}_E) \Big) \sum_{A\neq E} \frac{GM_A}{w_A^5} w_A^s w_A^j\,. \tag{50}$$

Returning to (30) and using (31) one gets

$$\hat{Q}_E^k = \hat{Q}_{E(N)}^k + \delta_I Q_E^k + \delta_D Q_E^k + \delta_t Q_E^k + \delta_p Q_E^k + c^{-2} \varepsilon_{lmn} v_E^m \omega_E^n v_E^s \big(\omega_E^s I_E^{lk} - \omega_E^k I_E^{ls} \big) + \\ + c^{-2} \sum_{A\neq E} \frac{GM_A}{r_{EA}^3} (\omega_E^m I_E^{sk} - \omega_E^k I_E^{sm} + \varepsilon_{jmn} \varepsilon_{kis} \omega_E^n I_E^{ij}) \times \\ \times \Big[(4v_A^m - 3v_E^m) r_{EA}^s - \tfrac{1}{2} (v_A^s r_{EA}^m + v_A^m r_{EA}^s) + \frac{3}{2r_{EA}^2} r_{EA}^s r_{EA}^m r_{EA}^l v_A^l \Big] + \\ + c^{-2} \sum_{A\neq E} GM_A \Big\{ \frac{3}{r_{EA}^5} \varepsilon_{kij} I_E^{is} r_{EA}^s r_{EA}^j \big[\tfrac{1}{2} (3v_E^2 + 3v_A^2 - 8 v_E^n v_A^n) - \bar{U}_A(\mathbf{x}_A) \big] + \\ + \tfrac{1}{2} \varepsilon_{kij} v_A^m v_E^s I_E^{in} \frac{\partial^4 r_{EA}}{\partial x_E^j \partial x_E^s \partial x_E^m \partial x_E^n} \Big\} + \tfrac{1}{2} c^{-2} \varepsilon_{kij} I_E^{js} \frac{\partial^2}{\partial x_E^i \partial x_E^s} \big(\bar{U}_E(\mathbf{x}_E) \big)^2\,. \tag{51}$$

Combination of (28), (38) and (47)–(49) results in

$$\delta_I Q_E^k + \delta_D Q_E^k + \delta_t Q_E^k + \delta_p Q_E^k + c^{-2}\varepsilon_{lmn} v_E^m \omega_E^n v_E^s \big(\omega_E^s I_E^{lk} - \omega_E^k I_E^{ls}\big) =$$
$$= c^{-2}\Big[\varepsilon_{kij} v_E^i v_E^m I_E^{jm}\omega_E^2 - \omega_E^k \varepsilon_{rmn} v_E^m \omega_E^n v_E^s I_E^{rs} +$$
$$+ v_E^i \omega_E^s \big(\varepsilon_{rij} v_E^s I_E^{rk}\omega_E^j - \tfrac12 \varepsilon_{kij} v_E^r I_E^{js}\omega_E^r - \tfrac12 \varepsilon_{kij} v_E^r I_E^{rs}\omega_E^j\big) +$$
$$+ \tfrac12 v_E^m v_E^n \big(I_E^{km}\dot\omega_E^n - I_E^{mn}\dot\omega_E^k - \varepsilon_{knj}\varepsilon_{mrs} I_E^{js}\dot\omega_E^r\big)\Big] +$$
$$+ 3c^{-2}\varepsilon_{ijk}\sum_{A\neq E}\frac{GM_A}{w_A^5}\Big[\big(\tfrac12 v_E^2 + 2\bar U_E(\mathbf{x}_E)\big) I_E^{is} w_A^j w_A^s + \tfrac12 I_E^{is} w_A^j w_A^s w_A^m a_E^m +$$
$$+ \tfrac12 I_E^{is} w_A^2 (w_A^j a_E^s + w_A^s a_E^j) - I_E^{is} v_E^m w_A^m (w_A^j v_A^s + w_A^s v_A^j) -$$
$$- \tfrac13 I_E^{jm} v_E^i v_E^m w_A^2 + \frac{5}{w_A^2} I_E^{is} w_A^j w_A^s v_E^m w_A^m \big(v_A^n - \tfrac12 v_E^n) w_A^n\Big] +$$
$$+ c^{-2} F^m \sum_{A\neq E}\frac{3GM_A}{w_A^5} w_A^s (I_E^{ks} w_A^m - I_E^{ms} w_A^k). \tag{52}$$

Substituting into (51) this expression together with

$$\frac{\partial^4 r_{EA}}{\partial x_E^j \partial x_E^s \partial x_E^m \partial x_E^n} = -\frac{1}{w_A^3}(\delta_{mn}\delta_{sj} + \delta_{ms}\delta_{nj} + \delta_{ns}\delta_{mj}) +$$
$$+ \frac{3}{w_A^5}\big(\delta_{mn} w_A^s w_A^j + \delta_{ms} w_A^n w_A^j + \delta_{ns} w_A^m w_A^j + \delta_{mj} w_A^n w_A^s +$$
$$+ \delta_{nj} w_A^m w_A^s + \delta_{sj} w_A^m w_A^n\big) - \frac{15}{w_A^7} w_A^m w_A^n w_A^s w_A^j$$

and

$$\frac{\partial^2}{\partial x_E^i \partial x_E^s}\big(\bar U_E(\mathbf{x}_E)\big)^2 = 2a_E^i a_E^s + 2\bar U_E(\mathbf{x}_E)\sum_{A\neq E}\frac{GM_A}{w_A^3}\left(-\delta_{is} + \frac{3}{w_A^2} w_A^i w_A^s\right)$$

one gets finally

$$\hat Q_E^k = \hat Q_{E(N)}^k + c^{-2}\Big[\varepsilon_{kij} v_E^i v_E^m I_E^{jm}\omega_E^2 - \omega_E^k \varepsilon_{rmn} v_E^m \omega_E^n v_E^s I_E^{rs} +$$
$$+ v_E^i \omega_E^s \big(\varepsilon_{rij} v_E^s I_E^{rk}\omega_E^j - \tfrac12 \varepsilon_{kij} v_E^r I_E^{js}\omega_E^r - \tfrac12 \varepsilon_{kij} v_E^r I_E^{rs}\omega_E^j\big) +$$
$$+ \tfrac12 v_E^m v_E^n \big(I_E^{km}\dot\omega_E^n - I_E^{mn}\dot\omega_E^k - \varepsilon_{knj}\varepsilon_{mrs} I_E^{js}\dot\omega_E^r\big)\Big] +$$
$$+ c^{-2}\sum_{A\neq E}\frac{GM_A}{r_{EA}^3}(\omega_E^m I_E^{sk} - \omega_E^k I_E^{sm} + \varepsilon_{jmn}\varepsilon_{kis}\omega_E^n I_E^{ij}) \times$$
$$\times \Big[(4v_A^m - 3v_E^m) r_{EA}^s - \tfrac12 (v_A^s r_{EA}^m + v_A^m r_{EA}^s) + \frac{3}{2r_{EA}^2} r_{EA}^s r_{EA}^m r_{EA}^l v_A^l\Big] +$$

$$
\begin{aligned}
&+c^{-2}\varepsilon_{kij}I_E^{is}\sum_{A\neq E}\frac{3GM_A}{w_A^5}\Big[\big(2v_E^2+\tfrac{3}{2}v_A^2-\tfrac{7}{2}v_E^m v_A^m-\bar U_A(\mathbf{x}_A)+\bar U_E(\mathbf{x}_E)\big)w_A^j w_A^s+\\
&+\tfrac{1}{3}(w_A^j a_E^s+w_A^s a_E^j)w_A^2+\tfrac{1}{2}w_A^j w_A^s w_A^m a_E^m-\tfrac{1}{6}(v_A^s v_E^j+v_A^j v_E^s)w_A^2+\\
&+\tfrac{1}{3}v_E^j v_E^s w_A^2-\tfrac{1}{2}v_E^m w_A^m\big(v_A^s w_A^j+v_A^j w_A^s\big)+\\
&+\tfrac{1}{2}v_A^m w_A^m(w_A^j v_E^s+w_A^s v_E^j)+\frac{5}{2w_A^2}(v_E^m w_A^m)(v_A^n-v_E^n)w_A^n w_A^j w_A^s\Big]+\\
&+c^{-2}F^m\sum_{A\neq E}\frac{3GM_A}{w_A^5}w_A^s(I_E^{ks}w_A^m-I_E^{ms}w_A^k).
\end{aligned}
\tag{53}
$$

From (19), (42) and (53) one has

$$
\begin{aligned}
N_E^i=\hat P_{ik}\Big\{&\hat Q^k_{E(N)}+c^{-2}\Big[\big(\hat\omega_E^k\hat I_E^{ms}-\hat\omega_E^m\hat I_E^{ks}\big)\hat\omega_E^s-\varepsilon_{kmn}\big(\dot{\hat\omega}_E^n\hat I_E^{ss}-\dot{\hat\omega}_E^s\hat I_E^{ns}\big)+\\
&+\sum_{A\neq E}\frac{3GM_A}{w_A^5}w_A^s(\hat I_E^{ks}w_A^m-\hat I_E^{ms}w_A^k)\Big]F^m+\\
&+\tfrac{1}{2}c^{-2}\Big(\varepsilon_{mkr}v_E^r v_E^s\hat\omega_E^2-\varepsilon_{mnr}v_E^r v_E^s\hat\omega_E^n\hat\omega_E^k+\varepsilon_{knr}v_E^m v_E^r\hat\omega_E^n\hat\omega_E^s-\varepsilon_{mkn}v_E^r v_E^s\hat\omega_E^r\hat\omega_E^n\Big)\hat I_E^{ms}+\\
&+c^{-2}\sum_{A\neq E}\frac{GM_A}{r_{EA}^3}\Big[\big(\varepsilon_{ink}\varepsilon_{smj}+\tfrac{1}{2}\varepsilon_{smn}\varepsilon_{ikj}+\varepsilon_{skm}\varepsilon_{inj}\big)\hat I_E^{is}\hat\omega_E^j+\\
&+\tfrac{1}{2}\hat I_E^{kn}\hat\omega_E^m-\tfrac{1}{2}\hat I_E^{km}\hat\omega_E^n\Big](4v_A^m-3v_E^m)r_{EA}^n+c^{-2}\varepsilon_{knm}\Big(\hat\omega_E^n\hat I_E^{ss}-\hat\omega_E^s\hat I_E^{ns}\Big)\dot F^m+\\
&+c^{-2}\varepsilon_{krj}\hat I_E^{rs}\sum_{A\neq E}\frac{3GM_A}{w_A^5}\Big[\big(2v_E^2+\tfrac{3}{2}v_A^2-\tfrac{7}{2}v_E^m v_A^m-\bar U_A(\mathbf{x}_A)+\bar U_E(\mathbf{x}_E)\big)w_A^j w_A^s+\\
&+\tfrac{1}{3}(w_A^j a_E^s+w_A^s a_E^j)w_A^2+\tfrac{1}{2}w_A^j w_A^s w_A^m a_E^m-\tfrac{1}{6}(v_A^s v_E^j+v_A^j v_E^s)w_A^2+\\
&+\tfrac{1}{3}v_E^j v_E^s w_A^2-\tfrac{1}{2}v_E^m w_A^m\big(v_A^s w_A^j+v_A^j w_A^s\big)+\\
&+\tfrac{1}{2}v_A^m w_A^m(w_A^j v_E^s+w_A^s v_E^j)+\frac{5}{2w_A^2}(v_E^m w_A^m)(v_A^n-v_E^n)w_A^n w_A^j w_A^s\Big].
\end{aligned}
\tag{54}
$$

In transforming to the GRS$^+$ quantities one has

$$
\begin{aligned}
N_E^i=N^i_{E(N)}&+c^{-2}\Big[\big(\hat\Omega_E^i\hat J_E^{ms}-\hat\Omega_E^m\hat J_E^{is}\big)\hat\Omega_E^s-\varepsilon_{imn}\big(\dot{\hat\Omega}_E^n\hat J_E^{ss}-\dot{\hat\Omega}_E^s\hat J_E^{ns}\big)+\\
&+\sum_{A\neq E}\frac{3GM_A}{w_A^5}W_A^s(\hat J_E^{is}W_A^m-\hat J_E^{ms}W_A^i)\Big]\mathcal{F}^m+\\
&+\tfrac{1}{2}c^{-2}\Big(\varepsilon_{mir}V_E^r V_E^s\Omega_E^2-\varepsilon_{mnr}V_E^r V_E^s\Omega_E^n\Omega_E^i+\varepsilon_{inr}V_E^m V_E^r\Omega_E^n\Omega_E^s-\varepsilon_{min}V_E^r V_E^s\Omega_E^r\Omega_E^n\Big)J_E^{ms}+\\
&+c^{-2}\sum_{A\neq E}{}^{(1)}\frac{GM_A}{w_A^3}\Big\{\Big[\big(\varepsilon_{sjm}\varepsilon_{kni}+\tfrac{1}{2}\varepsilon_{smn}\varepsilon_{kji}+\varepsilon_{sim}\varepsilon_{kjn}\big)\hat\Omega_E^j\hat J_E^{ks}+\\
&+\tfrac{1}{2}\hat\Omega_E^n\hat J_E^{im}-\tfrac{1}{2}\hat\Omega_E^m\hat J_E^{in}\Big](4V_A^m-3V_E^m)W_A^n+
\end{aligned}
$$

$$+\tfrac{1}{2}\left(\hat\Omega^n_E\hat J^{ss}_E-\hat\Omega^s_E\hat J^{ns}_E\right)\left[(4V^n_A-3V^n_E)W^i_A-(4V^i_A-3V^i_E)W^n_A\right]\Big\}+$$

$$+c^{-2}\varepsilon_{irj}\hat J^{rs}_E\sum_{A\neq E}^{(2)}\frac{3GM_A}{w^5_A}\Big[\big(2V^2_E+\tfrac{3}{2}V^2_A-\tfrac{7}{2}V^m_EV^m_A-\bar U_A(\mathbf{x}_A)+\bar U_E(\mathbf{x}_E)\big)W^j_AW^s_A+$$
$$+\tfrac{1}{3}(W^j_AA^s_E+W^s_AA^j_E)w^2_A+\tfrac{1}{2}W^j_AW^s_AW^m_AA^m_E-\tfrac{1}{6}(V^s_AV^j_E+V^j_AV^s_E)w^2_A+$$
$$+\tfrac{1}{3}V^j_EV^s_Ew^2_A-\tfrac{1}{2}V^m_EW^m_A\big(V^s_AW^j_A+V^j_AW^s_A\big)+$$
$$+\tfrac{1}{2}V^m_AW^m_A(W^j_AV^s_E+W^s_AV^j_E)+\frac{5}{2w^2_A}(V^m_EW^m_A)(V^n_A-V^n_E)W^n_AW^j_AW^s_A\Big] \qquad (55)$$

with

$$N^i_{E(N)}=3\varepsilon_{imn}\hat J^{ms}_E\sum_{A\neq E}\frac{GM_A}{w^5_A}W^s_AW^n_A+\dots\ . \qquad (56)$$

In addition to the GRS$^+$ quantities introduced above in (10), (11) and (45) the right-hand member (55) contains also

$$V^i_A=\hat P_{ik}v^k_A\,,\qquad V^i_E=\hat P_{ik}v^k_E\,,\qquad A^i_E=\hat P_{ik}a^k_E\,. \qquad (57)$$

It is easy to see that for the spherically symmetrical Earth with $\hat J^{ij}_E=\delta_{ij}\hat J_E$ the right-hand sides (55) vanish as stated in (Brumberg, 1995a). Moreover, the coefficient in $\mathcal{F}^m$ vanishes in virtue of equations (16) taken in the Newtonian approximation. The terms of order $c^{-2}Mq^4$ quadratic in the angular Earth velocity vanish as well. It may be easily checked that the terms of the order $c^{-2}Mq^4R/D$ staying under the sum $\sum^{(1)}$ mutually cancel. The terms of order $c^{-2}Mq^4R^2/D^2$ depending on the derivatives of the angular Earth velocity were cancelled in combining (42) and (53). The direct GRT perturbations in (55) are given by the terms of order $c^{-2}Mq^4R^2/D^2$ staying under the summation sign $\sum^{(2)}$. Hence, the right-hand member (55) consists of the Newtonian part (56) (needless to say that this Newtonian part of order Mq^2R^2/D^2 given here only in the quadrupole approximation should be computed in actual calculations with the whole necessary accuracy quite independently of relativistic terms) and the relativistic perturbations under the sign $\sum^{(2)}$. In result, the first of equations (16) takes the form

$$A\frac{d\hat\Omega^1_E}{du}+(C-B)\hat\Omega^2_E\hat\Omega^3_E=N^1_E \qquad (58)$$

with

$$N^1_E=N^1_{E(N)}+3(C-B)c^{-2}\sum_{A\neq E}\frac{GM_A}{w^5_A}\Big\{\Big[2V^2_E+\tfrac{3}{2}V^2_A-\tfrac{7}{2}V^m_EV^m_A-$$
$$-\bar U_A(\mathbf{x}_A)+\bar U_E(\mathbf{x}_E)+\tfrac{1}{2}W^m_AA^m_E+\frac{5}{2w^2_A}(V^m_EW^m_A)(V^n_A-V^n_E)W^n_A\Big]W^2_AW^3_A+$$
$$+\Big[\tfrac{1}{3}\big(W^2_AA^3_E+W^3_AA^2_E\big)-\tfrac{1}{6}\big(V^2_AV^3_E+V^3_AV^2_E\big)+\tfrac{1}{3}V^2_EV^3_E\Big]w^2_A+$$
$$+\tfrac{1}{2}V^m_AW^m_A\big(W^2_AV^3_E+W^3_AV^2_E\big)-\tfrac{1}{2}V^m_EW^m_A\big(V^2_AW^3_A+V^3_AW^2_A\big)\Big\}, \qquad (59)$$

$$N^1_{E(N)} = 3(C-B)\sum_{A\neq E}\frac{GM_A}{w_A^5}W_A^2W_A^3 + \ldots \tag{60}$$

and

$$\bar{U}_A(\mathbf{x}_A) = \sum_{B\neq A}\frac{GM_B}{r_{AB}}\,, \qquad \bar{U}_E(\mathbf{x}_E) = \sum_{A\neq E}\frac{GM_A}{r_{EA}}\,. \tag{61}$$

Two other equations of (16) are obtained by the circular permutation of indices 1, 2 and 3 and letters A, B, C denoting the Earth principal inertia moments (not to be mixed with letters A and B designating celestial bodies such as Earth (E), Sun (S) and Moon (M)).

Equations (58) are obtained here only up to the order $c^{-2}Mq^4R^2/D^2$ inclusively in the relativistic right-hand members taking into account only quadrupole inertia moments of the Earth and treating the Sun and the Moon as point masses. This seems to be quite sufficient for the present day applications. The technique employed in deriving these equations enables one easily to modify them. For example, if one still prefers to deal with TCB it is necessary to remove the correction (50) from the right-hand members starting with (53).

Conclusion

This paper may be regarded as an up-to-date version of the old results by the author (Brumberg, 1962, 1972). The equations given there were written in terms of the BRS quantities. The contributions due to the Earth's pressure were given only in the integral form and actually were neglected in the final equations. In particular, this is the reason of the appearance in the right-hand members of a large term of order $c^{-2}Mq^4$. In the present paper this term occurs in (30) and then cancels due to $\delta_p Q^k_E$ correction in (52). Pushkarev and Abdil'din (1976) demonstrated the importance of the Earth's pressure contributions removing this term from the equations of rotation of the spherically symmetric Earth. Written, as before, in terms of the BRS quantities their equations still contain a lot of non-physical terms (in particular, spherical symmetry was considered by them just in BRS). The present paper improves the results (Brumberg, 1968, 1972; Pushkarev and Abdil'din, 1976) by more rigorous treatment of the pressure contributions and by expressing the final equations in terms of the GRS quantities (including the GRS framework for rigid body velocity distribution, Earth's moments of inertia and angular velocity). Of course, it is desirable to derive the GRT Earth's rotation equations by some other technique and to compare the final results. At the same time. the equations given here may be already directly applied to take into account the GRT corrections in the practical analysis of the Earth's rotation problem.

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