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ALGORITHMS OF RELATIVISTIC REDUCTION OF SPACE ASTROMETRIC OBSERVATIONS

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Abstract.

The paper presents a slight modification of algorithms exposed in (Brumberg, 1991; Bureau des Longitudes, 1997) for relativistic reduction of high precision space astrometry observations.

1. Introduction

In anticipation of realization of space astrometry projects intended for microarcsecond accuracy it seems reasonable to specify algorithms of relativistic reduction of such observations. Just as in Newtonian astrometry there might be various options to treat this problem. This paper is aimed to develop and improve the algoritms exposed earlier in (Brumberg, 1991; Bureau des Longitudes, 1997) and slightly different from those of (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992). In doing it we've tried to retain the universal form of the algorithms as much as possible (to facilitate the consideration of different perturbing effects in the light propagation) avoiding at the same time any specific general relativistic techniques.

2. Hierarchy of Reference Systems

As a starting point we will use the hierarchy of relativistic reference systems (RS) involving barycentric (BRS), geocentric (GRS) and satellite (SRS) reference systems as exposed, for instance, in Section 2 of (Brumberg, 1995). This hierarchy may be illustrated as follows:

$$\mathrm{BRS} \longrightarrow \begin{cases} \mathrm{DGRS} \quad (q=1) \longrightarrow \begin{cases} \mathrm{SRS1} \quad (\hat{q}=1, \quad \tilde{q}=1) \\ \mathrm{SRS2} \quad (\hat{q}=0, \quad \tilde{q}=1) \end{cases} \\ \\ \mathrm{KGRS} \quad (q=0) \longrightarrow \begin{cases} \mathrm{SRS3} \quad (\hat{q}=1, \quad \tilde{q}=1) \\ \mathrm{SRS4} \quad (\hat{q}=0, \quad \tilde{q}=1) \\ \mathrm{SRS5} \quad (\hat{q}=0, \quad \tilde{q}=0) \end{cases} \end{cases} \qquad (2.1)$$

A single RS at the barycenric level with some given orientation of the spatial axes (BRS) generates at the geocentric level two different systems, dynamically nonrotating system (DGRS) and kinematically nonrotating system (KGRS). One may treat these systems as one system supplied by numerical parameter q taking values 1 or 0, correspondingly. In its turn each of this system generates at the satellite level (related to a satellite orbiting the Earth) two systems, dynamically (DSRS) or kinematically (KSRS) nonrotating with respect to the generating GRS. One may again distinguish these systems by values 1 or 0 of numerical parameter $\hat{q}$. Transformations BRS $\to$ GRS and GRS $\to$ SRS represent generalized Lorentz transformations. Even in case of special relativity two consequent Lorentz transformations without rotation (BRS $\to$ KGRS and KGRS $\to$ SRS4) result in spatial rotation of the final system (SRS4) with respect to the initial one (BRS). The expression for this rotation in the general relativity case was derived in (Klioner, 1993). Consideration of this rotation leads to the fifth system at the satellite level, SRS5, kinematically nonrotating with respect to BRS. To describe all five satellite systems as one system one has to introduce additive numerical parameter $\tilde{q}$ equal to 1 for all four preceding systems and vanishing for SRS5.

3. Measurable vs. Coordinate Light Direction

Any relativistic RS may be described by metric of the type

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \qquad x^0 = ct \,, \tag{3.1}$$

greek indices running values from 0 to 3 with summation over repeating indices. Even in case of Newtonian rotation of the spatial axes coefficients $g_{\mu\nu}$ are supposed to differ from their Minkowski values $\eta_{\mu\nu}$ (special relativity flat space–time) by small corrections $h_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{3.2}$$

$$\eta_{00} = 1 \,, \qquad \eta_{0i} = 0 \,, \qquad \eta_{ij} = \delta_{ij} \,, \tag{3.3}$$

$$h_{00} \,, h_{ij} \sim \mathrm{O}(c^{-2}), \qquad h_{0i} \sim \begin{cases} \mathrm{O}(c^{-1}) & \text{for} \quad \text{rotating} \quad \mathrm{RS}\,, \\ \mathrm{O}(c^{-3}) & \text{for} \quad \text{nonrotating} \quad \mathrm{RS}\,, \end{cases}$$

latin indices running values from 1 to 3. Four–dimensional quadratic form (3.1) may be reduced locally to the algebraic sum of squares. At first, one has

$$ds^2 = c^2 d\tau^2 - dl^2 \tag{3.4}$$

with

$$d\tau = \frac{1}{\sqrt{g_{00}}} g_{0\alpha} dx^\alpha \tag{3.5}$$

and

$$dl^2 = \gamma_{ik} dx^i dx^k \tag{3.6}$$

where

$$\gamma_{ik} = \frac{1}{g_{00}} g_{0i} g_{0k} - g_{ik} \,. \tag{3.7}$$

By substituting (3.2) into (3.5) and (3.7) one yields

$$d\tau = (1 + h_{00})^{1/2} c dt + (1 + h_{00})^{-1/2} h_{0i} dx^i \tag{3.8}$$

and

$$\gamma_{ik} = \delta_{ik} - h_{ik} + (1 + h_{00})^{-1} h_{0i} h_{0k} \,. \tag{3.9}$$

The three–dimensional quadratic form (3.6) may be easily reduced to the sum of squares

$$dl^2 = \delta_{ik} dx^{(i)} dx^{(k)} \tag{3.10}$$

by linear transformation

$$dx^{(i)} = dx^i + \lambda_{ij} dx^j \tag{3.11}$$

with symmetrical coefficients λ_{ij} to be determined from the equations

$$2\lambda_{ik} + \lambda_{mi}\lambda_{mk} = -h_{ik} + (1 + h_{00})^{-1} h_{0i} h_{0k} \,. \tag{3.12}$$

Metric (3.4) with (3.10) presented locally as Minkowski metric enables one to find the measurable light direction in the form

$$p^{(i)} = c^{-1} \frac{dx^{(i)}}{d\tau} \,. \tag{3.13}$$

These components are to be compared with the components $\dot{x}^i$ of the coordinate light velocity in the field (3.1). Using (3.8) and (3.13) one obtains

$$p^{(i)} = \frac{c^{-1}(\dot{x}^i + \lambda_{ij}\dot{x}^j)}{(1 + h_{00})^{1/2} + c^{-1}(1 + h_{00})^{-1/2} h_{0k}\dot{x}^k} \tag{3.14}$$

and

$$c^{-1}\dot{x}^i = \frac{p^{(i)} - \lambda_{ij} p^{(j)}}{(1 + h_{00})^{-1/2} - (1 + h_{00})^{-1} h_{0k} p^{(k)}} \,. \tag{3.15}$$

Until now all the above formulas are rigorous. The approximate solution of (3.12) results in

$$\begin{aligned} \lambda_{ik} = & - \tfrac{1}{2} h_{ik} + \tfrac{1}{2} h_{0i} h_{0k} - \tfrac{1}{8} h_{im} h_{km} - \tfrac{1}{2} h_{00} h_{0i} h_{0k} + \\ & + \tfrac{1}{8}(h_{0i} h_{km} + h_{0k} h_{im}) h_{0m} - \tfrac{1}{8} h_{0i} h_{0k} h_{0m} h_{0m} + \mathrm{O}(c^{-6}). \end{aligned} \tag{3.16}$$

In case of nonrotating RS all terms containing h_{0i} should be omitted. For nonrotating systems in harmonic coordinates within the post-Newtonian approximation one has

$$h_{ij} = \delta_{ij} h_{00} \tag{3.17}$$

and

$$\lambda_{ik} = -\tfrac{1}{2} \delta_{ik} h_{00} + \mathrm{O}(c^{-4}), \tag{3.18}$$

$$p^{(i)} = (1 - h_{00}) c^{-1} \dot{x}^i + \mathrm{O}(c^{-4}), \tag{3.19}$$

$$c^{-1}\dot{x}^i = (1 + h_{00}) p^{(i)} + \mathrm{O}(c^{-4}). \tag{3.20}$$

4. General Reduction

First of all, to distinguish between BRS, GRS and SRS quantities let us mark GRS and SRS quantities by hat and tilde, respectively.

The starting point in the general reduction technique developed in (Brumberg, 1991) is the solution of the BRS equations of the light propagation. This solution may be presented in the form

$$\mathbf{x}(t) = \mathbf{x}_0 + c(t - t_0)\boldsymbol{\sigma} + \Delta\mathbf{x}\,, \tag{4.1}$$

$$\dot{\mathbf{x}}(t) = c\boldsymbol{\sigma} + \Delta\dot{\mathbf{x}}\,, \tag{4.2}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0\,, \qquad \dot{\mathbf{x}}(-\infty) = c\boldsymbol{\sigma}\,, \qquad \boldsymbol{\sigma}^2 = 1\,, \tag{4.3}$$

$\mathbf{x}_0$ and $\boldsymbol{\sigma}$ being two arbitrary vectorial constants. Here $\Delta\mathbf{x}$ and $\Delta\dot{\mathbf{x}}$ stand for general relativity terms. Their expressions may be found in (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992) but one may use any other suitable expressions for these quantities as well. The technique exposed below does not demand their explicit expressions. As exposed in Section 3 one may relate BRS coordinate light velocity $\dot{x}^i$ and BRS measurable light direction $p^{(i)}$ resulting in virtue of (4.2) in the BRS $(t\,, x^i)$ reduction formula:

$$v^i = \frac{dx^i}{dt}\,, \qquad p^{(i)} = \sigma^i + \delta p^{(i)}\,. \tag{4.4}$$

In the same manner the technique of Section 3 enables one to relate GRS coordinate light velocity $\hat{v}^i$ and GRS measurable light direction $\hat{p}^{(i)}$. On the other hand, BRS→GRS transformation involves the relationship between v^i and $\hat{v}^i$. In result, one gets the GRS $(\hat{t}\,, \hat{x}^i\,, q)$ reduction formula:

$$\hat{v}^i = \frac{d\hat{x}^i}{d\hat{t}} = v^i + \delta\hat{v}^i\,, \qquad \hat{p}^{(i)} = p^{(i)} + \delta\hat{p}^{(i)}\,. \tag{4.5}$$

Repeating this procedure once more one relate SRS coordinate light velocity $\tilde{v}^i$ and SRS measurable light direction $\tilde{p}^{(i)}$. On the other hand, GRS→SRS transformation involves the relationship between $\hat{v}^i$ and $\tilde{v}^i$. In result, one gets the SRS $(\tilde{t}\,, \tilde{x}^i\,, \hat{q}\,, \tilde{q})$ reduction formula:

$$\tilde{v}^i = \frac{d\tilde{x}^i}{d\tilde{t}} = \hat{v}^i + \delta\tilde{v}^i\,, \qquad \tilde{p}^{(i)} = \hat{p}(i) + \delta\tilde{p}^{(i)}\,. \tag{4.6}$$

Combination of (4.4)–(4.6) leads to the final reduction formula:

$$\tilde{p}^{(i)} = \sigma^i + \delta p^{(i)} + \delta\hat{p}^{(i)} + \delta\tilde{p}^{(i)} \tag{4.7}$$

relating the actually observed light direction $\tilde{p}^{(i)}$ and the coordinate light direction σ^i. In what follows we will give correction terms $\delta p^{(i)}$, $\delta\hat{p}^{(i)}$, $\delta\tilde{p}^{(i)}$ as well as auxiliary corrections $\delta\hat{v}^i$, $\delta\tilde{v}^i$. The rigorous expressions (3.12), (3.14), (3.15) enable one to compute these quantities within the accuracy of BRS→GRS and GRS→SRS transformations. To have simple analytical formulas for fast evaluation of the relativistic effects we give below the post–Newtonian expressions based on (3.18)–(3.20).

An equivalent reduction formula slightly different from (4.7) by its form is presented in (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992).

Formula (4.7) is directly valid for observations from an Earth's artificial satellite. It may be easily modified for other cases as, for instance, observations from an interplanetary probe. In the latter case it is sufficient to do only one transformation from BRS to the system related to such probe.

5. BRS Reduction

Rewritting (4.2) in the form

$$c^{-1}\dot{x}^i = \sigma^i + c^{-1}\Delta\dot{x}^i \tag{5.1}$$

one finds the relativistic term of the BRS reduction formula (4.4)

$$\delta p^{(i)} = -h_{00}\sigma^i + c^{-1}\Delta\dot{x}^i\,. \tag{5.2}$$

If necessary, one should transform σ^i to take into account parallax and proper motion corrections. This question is considered below in Section 8.

6. GRS Reduction

Applied to GRS quantities the equation (3.19) involves

$$\hat{p}^{(i)} = (1-\hat{h}_{00})c^{-1}\hat{v}^i \tag{6.1}$$

and then

$$\hat{p}^{(i)} = (1+h_{00}-\hat{h}_{00})p^{(i)} + c^{-1}\delta\hat{v}^i\,. \tag{6.2}$$

BRS→GRS transformation involves (Brumberg, 1995a)

$$\begin{aligned}\delta\hat{v}^i =& - v_E^i + (c^{-1}v^k v_E^k)c^{-1}v^i + c^{-1}[(c^{-1}v^k v_E^k)^2 c^{-1}v^i - \tfrac{1}{2}c^{-1}v^i v_E^2 - \\ &- \tfrac{1}{2}v_E^i(c^{-1}v^k v_E^k) + (qF^{ik} + 2D^{ik} + 2D^{ikm}\hat{x}^m)c^{-1}v^k + a_E^k\hat{x}^k c^{-1}v^i].\end{aligned} \tag{6.3}$$

By substituting this expression into (6.2) one gets

$$\begin{aligned}\hat{p}^{(i)} =& (1+h_{00}-\hat{h}_{00})p^{(i)} + c^{-1}[\mathbf{p}\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} + c^{-2}(\mathbf{p}\mathbf{v}_E)[\mathbf{p}\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} - \\ &- \tfrac{1}{2}c^{-2}[\mathbf{v}_E\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} + c^{-2}(qF^{ik}+2D^{ik}+2D^{ikm}\hat{x}^m)p^{(k)} + \\ &+ c^{-2}a_E^k\hat{x}^k p^{(i)}\,.\end{aligned} \tag{6.4}$$

In virtue of the relation

$$h_{00} - \hat{h}_{00} = -2\bar{U}_E(\mathbf{x}_E) - 2a_E^k\hat{x}^k \tag{6.5}$$

one obtains the GRS reduction formula

$$\begin{aligned}\delta\hat{p}^{(i)} =& c^{-1}[\mathbf{p}\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} + c^{-2}(\mathbf{p}\mathbf{v}_E)[\mathbf{p}\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} - \\ &- \tfrac{1}{2}c^{-2}[\mathbf{v}_E\times(\mathbf{p}\times\mathbf{v}_E)]^{(i)} + c^{-2}qF^{ik}p^{(k)} + c^{-2}(a_E^k\hat{x}^i - a_E^i\hat{x}^k)p^{(k)}\,.\end{aligned} \tag{6.6}$$

In (6.4) we have used

$$D^{ik}(t) = \delta_{ik}\bar{U}_E(\mathbf{x}_E) \tag{6.7}$$

and

$$D^{ikm}(t) = \tfrac{1}{2}(\delta_{ik}a_E^m + \delta_{im}a_E^k - \delta_{km}a_E^i) \tag{6.8}$$

where $\bar{U}_E(\mathbf{x})$ stands for the Newtonian potential of all solar system bodies excepting the Earth, x_E^i, v_E^i and a_E^i being Earth's BRS position, velocity and acceleration, respectively, with

$$a_E^i = \bar{U}_{E,i}(\mathbf{x}_E) - Q_i\,, \tag{6.9}$$

Q_i being nongeodesic acceleration in the Earth's BRS motion. Besides,

$$\dot{F}^{ij} = \tfrac{3}{2}(v_E^i a_E^j - v_E^j a_E^i) - 2\big[\bar{U}^i_{E,j}(\mathbf{x}_E) - \bar{U}^j_{E,i}(\mathbf{x}_E)\big] + 2(v_E^i Q_j - v_E^j Q_i), \tag{6.10}$$

$\bar{U}_E^i(\mathbf{x})$ denoting the Newtonian vector–potential of all solar system bodies excepting the Earth. Comma in (6.9) and (6.10) denotes the partial derivative with respect to the variable separated by comma.

7. SRS Reduction

By applying now (3.19) to SRS one has

$$\tilde{p}^{(i)} = (1 - \tilde{h}_{00})c^{-1}\tilde{v}^i \tag{7.1}$$

and then

$$\tilde{p}^{(i)} = (1 + \hat{h}_{00} - \tilde{h}_{00})\hat{p}^{(i)} + c^{-1}\delta\tilde{v}^i\,. \tag{7.2}$$

GRS→SRS transformation involves (Brumberg, 1995a)

$$\begin{aligned}\delta\tilde{v}^i =& -\hat{v}_S^i + (c^{-1}\hat{v}^k\hat{v}_S^k)c^{-1}\hat{v}^i + c^{-1}\big\{(c^{-1}\hat{v}^k\hat{v}_S^k)^2 c^{-1}\hat{v}^i - \tfrac{1}{2}c^{-1}\hat{v}^i\hat{v}_S^2 - \\ &- \tfrac{1}{2}\hat{v}_S^i(c^{-1}\hat{v}^k\hat{v}_S^k) + \big[\hat{q}R^{ik} + (\tilde{q}-1)K^{ik} + 2\mathcal{D}^{ik} + 2\mathcal{D}^{ikm}\tilde{x}^m\big]c^{-1}\hat{v}^k + \\ &+ \hat{a}_S^k\tilde{x}^k c^{-1}\hat{v}^i\big\}.\end{aligned} \tag{7.3}$$

Combining (7.2) and (7.3) one gets

$$\begin{aligned}\tilde{p}^{(i)} =& (1 + \hat{h}_{00} - \tilde{h}_{00})\hat{p}^{(i)} + c^{-1}[\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \hat{\mathbf{v}}_S)]^{(i)} + c^{-2}(\hat{\mathbf{p}}\hat{\mathbf{v}}_S)[\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \hat{\mathbf{v}}_S)]^{(i)} - \\ &- \tfrac{1}{2}c^{-2}\big[\hat{\mathbf{v}}_S \times (\hat{\mathbf{p}} \times \hat{\mathbf{v}}_S)\big]^{(i)} + c^{-2}\big[\hat{q}R^{ik} + (\tilde{q}-1)K^{ik} + 2\mathcal{D}^{ik} + \\ &+ 2\mathcal{D}^{ikm}\tilde{x}^m\big]\hat{p}^{(k)} + c^{-2}\hat{a}_S^k\tilde{x}^k\hat{p}^{(i)}.\end{aligned} \tag{7.4}$$

By means of the relation

$$\hat{h}_{00} - \tilde{h}_{00} = -2\big[\hat{U}_E(\hat{\mathbf{x}}_S) + Q_k\hat{x}_S^k + T(\hat{\mathbf{x}}_S)\big] - 2\hat{a}_S^k\tilde{x}^k \tag{7.5}$$

one gets the SRS reduction formula

$$\begin{aligned}\delta\tilde{p}^{(i)} =& c^{-1}[\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \hat{\mathbf{v}}_S)]^{(i)} + c^{-2}(\hat{\mathbf{p}}\hat{\mathbf{v}}_S)[\hat{\mathbf{p}} \times (\hat{\mathbf{p}} \times \hat{\mathbf{v}}_S)]^{(i)} - \\ &- \tfrac{1}{2}c^{-2}[\hat{\mathbf{v}}_S \times (\hat{\mathbf{p}} \times \hat{\mathbf{v}}_S)]^{(i)} + c^{-2}\big[\hat{q}R^{ik} + (\tilde{q}-1)K^{ik}\big]\hat{p}^{(k)} + \\ &+ c^{-2}(\hat{a}_S^k\tilde{x}^i - \hat{a}_S^i\tilde{x}^k)\hat{p}^{(k)}\,.\end{aligned} \tag{7.6}$$

In (7.4) one meets the coefficients

$$\mathcal{D}^{ik}(\hat{t}) = \left[\hat{U}_E(\hat{\mathbf{x}}_S) + Q_m \hat{x}^m_S + T(\hat{\mathbf{x}}_S)\right] \tag{7.7}$$

and

$$\mathcal{D}^{ikm}(\hat{t}) = \tfrac{1}{2}(\delta_{ik}\hat{a}^m_S + \delta_{im}\hat{a}^k_S - \delta_{km}\hat{a}^i_S) \tag{7.8}$$

with the GRS tidal potential

$$T(\hat{\mathbf{x}}) = \bar{U}_E(\mathbf{x}_E + \hat{\mathbf{x}}) - \bar{U}_E(\mathbf{x}_E) - \bar{U}_{E,j}(\mathbf{x}_E)\hat{x}^j \tag{7.9}$$

and geocentric satellite acceleration $\hat{a}^i_S$. Introducing the quantity

$$E_i = -\hat{a}^i_S + \hat{U}_{E,i}(\hat{\mathbf{x}}_S) + Q_i + \bar{U}_{E,i}(\mathbf{x}_E + \hat{\mathbf{x}}_S) - \bar{U}_{E,i}(\mathbf{x}_E) \tag{7.10}$$

one may present the $\hat{t}$–derivative of the topocentric–type precession in the form

$$\begin{aligned}\dot{R}_{ij} =& \tfrac{3}{2}(\hat{v}^i_S\hat{a}^j_S - \hat{v}^j_S\hat{a}^i_S) + (\dot{a}^i_E\hat{x}^j_S - \dot{a}^j_E\hat{x}^i_S) - 2[\hat{U}^i_{E,j}(\hat{\mathbf{x}}_S) - \hat{U}^j_{E,i}(\hat{\mathbf{x}}_S)] + \\ &+ 2[v^i_E\bar{U}_{E,jk}(\mathbf{x}_E) - v^j_E\bar{U}_{E,ik}(\mathbf{x}_E) - \bar{U}^i_{E,jk}(\mathbf{x}_E) + \bar{U}^j_{E,ik}(\mathbf{x}_E)]\hat{x}^k_S + \\ &+ 2(\hat{v}^i_S E_j - \hat{v}^j_S E_i),\end{aligned} \tag{7.11}$$

$\hat{U}_E(\hat{\mathbf{x}})$ and $\hat{U}^i_E(\hat{\mathbf{x}})$ being the GRS geopotential and vector–geopotential. For SRS the quantity E_i vanishes resulting to the Newtonian GRS satellite equations of motion. If S denotes a point on the surface of the Earth then E_i does not vanish and the right–hand of (7.11) relates just to the topocentric precession. The purely kinematical precession caused by two subsequent BRS→GRS and GRS→SRS transformations (Klioner, 1993) is given by

$$K^{ij} = \hat{x}^i_S a^j_E - \hat{x}^j_S a^i_E + \tfrac{1}{2}(\hat{v}^i_S v^j_E - \hat{v}^j_S v^i_E). \tag{7.12}$$

8. Parallax and Proper Motion

Corrections for parallax and proper motion may be taken just as in Newtonian astrometry. We reproduce in this section the derivation in vectorial form of (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992). These corrections are related to the boundary problem of light propagation. Denoting

$$\mathbf{R}(t\,, t_0) = \mathbf{x} - \mathbf{x}_0 \tag{8.1}$$

one may rewrite (4.1) in the form

$$c(t - t_0)\boldsymbol{\sigma} = \mathbf{R} - \Delta\mathbf{x}\,, \qquad \Delta\mathbf{x}(t_0) = 0\,. \tag{8.2}$$

Hence,

$$c(t - t_0) = R\left[1 - \frac{2}{R^2}\mathbf{R}\Delta\mathbf{x} + \frac{1}{R^2}(\Delta\mathbf{x})^2\right]^{1/2}. \tag{8.3}$$

Introducing the unit vector directed to the light source

$$\mathbf{k} = -\frac{\mathbf{R}}{R} \tag{8.4}$$

one gets from (8.2) and (8.3)

$$c(t - t_0) = R\left[1 + \frac{1}{R}\mathbf{k}\Delta\mathbf{x} + \frac{1}{2R^2}(\Delta\mathbf{x}\times\mathbf{k})^2 + \ldots\right] \tag{8.5}$$

and

$$\begin{aligned}\boldsymbol{\sigma} =& -\mathbf{k} - \frac{1}{R}[\mathbf{k}\times(\Delta\mathbf{x}\times\mathbf{k})] + \frac{1}{2R^2}(\Delta\mathbf{x}\times\mathbf{k})^2\mathbf{k} + \\ &+ \frac{1}{R^2}(\mathbf{k}\Delta\mathbf{x})[\mathbf{k}\times(\Delta\mathbf{x}\times\mathbf{k})] + \ldots .\end{aligned} \tag{8.6}$$

It is appropriate to remind here the leading relativistic terms in light propagation (Brumberg, 1991)

$$\Delta\mathbf{x} = 2\sum_A m_A\left[\frac{\boldsymbol{\sigma}\times(\mathbf{r}_{0A}\times\boldsymbol{\sigma})}{r_{0A} - \boldsymbol{\sigma}\mathbf{r}_{0A}} - \frac{\boldsymbol{\sigma}\times(\mathbf{r}_A\times\boldsymbol{\sigma})}{r_A - \boldsymbol{\sigma}\mathbf{r}_A} - \boldsymbol{\sigma}\ln\frac{r_A + \boldsymbol{\sigma}\mathbf{r}_A}{r_{0A} + \boldsymbol{\sigma}\mathbf{r}_{0A}}\right] \tag{8.7}$$

and

$$\Delta\dot{\mathbf{x}} = -2c\sum_A \frac{m_A}{r_A}\left[\boldsymbol{\sigma} + \frac{\boldsymbol{\sigma}\times(\mathbf{r}_A\times\boldsymbol{\sigma})}{r_A - \boldsymbol{\sigma}\mathbf{r}_A}\right]. \tag{8.8}$$

Here

$$m_A = \frac{GM_A}{c^2}\,, \qquad \mathbf{r}_A = \mathbf{x} - \mathbf{x}_A\,, \qquad \mathbf{r}_{0A} = \mathbf{x}_0 - \mathbf{x}_A\,, \tag{8.9}$$

M_A being mass of body A. Summation in (8.7) and (8.8) is performed over all solar system bodies marked by capital latin letters. Relativistic terms due to the nonsphericity of the bodies and their rotation as well as the second order monopole terms (post–post–Newtonian terms) may be added, by example, from (Brumberg *et al.*, 1990; Klioner and Kopeikin, 1992) or elsewhere. Correction for parallax is introduced under the condition

$$|\mathbf{x}| \ll |\mathbf{x}_0| \equiv \rho\,. \tag{8.10}$$

In this case

$$\begin{aligned}\mathbf{k} =& \frac{\mathbf{x}_0}{\rho} - \frac{1}{\rho^3}[\mathbf{x}_0\times(\mathbf{x}\times\mathbf{x}_0)] - \frac{1}{2\rho^5}(\mathbf{x}\times\mathbf{x}_0)^2\mathbf{x}_0 - \\ &- \frac{1}{\rho^5}(\mathbf{x}\mathbf{x}_0)[\mathbf{x}_0\times(\mathbf{x}\times\mathbf{x}_0)] + \ldots .\end{aligned} \tag{8.11}$$

Proper motion correction is introduced to take into account the time interval between the initial epoch of emission t_0^* and the moment t_0 of the light emission. One has therewith

$$\mathbf{x}_0(t_0) = \mathbf{x}_0^* + \dot{\mathbf{x}}_0^*\Delta t_0 + \tfrac{1}{2}\ddot{\mathbf{x}}_0^*(\Delta t_0)^2 + \ldots \tag{8.12}$$

with

$$\Delta t_0 = t_0 - t_0^*\,, \qquad \mathbf{x}_0^* = \mathbf{x}_0(t_0^*), \qquad \dot{\mathbf{x}}_0^* = \dot{\mathbf{x}}_0(t_0^*), \qquad \ddot{\mathbf{x}}_0^* = \ddot{\mathbf{x}}_0(t_0^*). \tag{8.13}$$

Putting

$$\rho^* = |\mathbf{x}_0^*|, \qquad \mathbf{k}_0 = \frac{\mathbf{x}_0^*}{\rho^*} \tag{8.14}$$

and using auxiliary expansions

$$\frac{1}{\rho} = \frac{1}{\rho^*}\Big\{1 - \frac{1}{\rho^*}\mathbf{k}_0\dot{\mathbf{x}}_0^*\Delta t_0 + \frac{1}{2\rho^*}\Big[\frac{3}{\rho^*}(\mathbf{k}_0\dot{\mathbf{x}}_0^*)^2 - \frac{1}{\rho^*}(\dot{\mathbf{x}}_0^*)^2 - \mathbf{k}_0\ddot{\mathbf{x}}_0^*\Big](\Delta t_0)^2 + \ldots\Big\},$$

and

$$\begin{aligned}\frac{\mathbf{x}_0}{\rho} =&\mathbf{k}_0 + \frac{1}{\rho^*}[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)]\Delta t_0 + \frac{1}{\rho^*}\Big\{\tfrac{1}{2}[\mathbf{k}_0 \times (\ddot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \\ &- \frac{1}{\rho^*}(\mathbf{k}_0\dot{\mathbf{x}}_0^*)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \frac{1}{2\rho^*}(\mathbf{k}_0 \times \dot{\mathbf{x}}_0^*)^2\mathbf{k}_0\Big\}(\Delta t_0)^2 + \ldots\end{aligned}$$

one gets

$$\begin{aligned}\mathbf{k} =&\mathbf{k}_0 - \frac{1}{\rho^*}[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] - \frac{1}{2\rho^{*2}}(\mathbf{x} \times \mathbf{k}_0)^2\mathbf{k}_0 - \frac{1}{\rho^{*2}}(\mathbf{x}\mathbf{k}_0)[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] + \\ &+ \frac{1}{\rho^*}\Big\{[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] + \frac{1}{\rho^*}(\mathbf{x}\mathbf{k}_0)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] + \\ &+ \frac{1}{\rho^*}(\dot{\mathbf{x}}_0^*\mathbf{k}_0)[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] + \frac{1}{\rho^*}\mathbf{k}_0\big(\mathbf{x}[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)]\big)\Big\}\Delta t_0 + \\ &+ \frac{1}{\rho^*}\Big\{\tfrac{1}{2}[\mathbf{k}_0 \times (\ddot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \frac{1}{\rho^*}(\dot{\mathbf{x}}_0^*\mathbf{k}_0)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \\ &- \frac{1}{2\rho^*}(\mathbf{x} \times \mathbf{k}_0^*)^2\mathbf{k}_0\Big\}(\Delta t_0)^2 + \ldots\,.\end{aligned} \tag{8.15}$$

Introducing now the vector of parallax

$$\boldsymbol{\pi} = \frac{1}{\rho^*}[\mathbf{k}_0 \times (\mathbf{x} \times \mathbf{k}_0)] \tag{8.16}$$

and vector of proper motion

$$\boldsymbol{\mu} = \mathbf{k}_0 \times (\dot{\mathbf{k}}_0 \times \mathbf{k}_0) \tag{8.17}$$

and using the evident relations

$$\mathbf{k}_0\boldsymbol{\pi} = 0\,, \qquad \mathbf{k}_0\boldsymbol{\mu} = 0$$

as well as the derivatives

$$\dot{\mathbf{k}}_0 = \frac{1}{\rho^*}[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)] \tag{8.18}$$

and

$$\dot{\boldsymbol{\mu}} = \frac{1}{\rho^*}\Big\{[\mathbf{k}_0 \times (\ddot{\mathbf{x}}_0^* \times \mathbf{k}_0)] - \frac{1}{\rho^*}(\mathbf{k}_0 \times \dot{\mathbf{x}}_0^*)^2\mathbf{k}_0 - \frac{2}{\rho^*}(\mathbf{k}_0\dot{\mathbf{x}}_0^*)[\mathbf{k}_0 \times (\dot{\mathbf{x}}_0^* \times \mathbf{k}_0)]\Big\} \tag{8.19}$$

one may present (8.15) in the more compressed form

$$\mathbf{k} = \big(1 + \boldsymbol{\pi}\boldsymbol{\mu}\Delta t_0 - \tfrac{1}{2}\boldsymbol{\pi}^2\big)\mathbf{k}_0 + \big(1 + \frac{1}{\rho^*}\mathbf{x}\mathbf{k}_0\big)\boldsymbol{\mu}\Delta t_0 - \\ - \big(1 + \frac{1}{\rho^*}\mathbf{x}\mathbf{k}_0 - \frac{1}{\rho^*}\dot{\mathbf{x}}_0^*\mathbf{k}_0\Delta t_0\big)\boldsymbol{\pi} + \tfrac{1}{2}\dot{\boldsymbol{\mu}}(\Delta t_0)^2 + \dots . \tag{8.18}$$

It remains to eliminate the unmeasurable time interval Δt_0 at the point of the light emission by means of the transformation $\Delta t_0 \to \Delta t = t - t^*$, t^* being the BRS moment corresponding to t_0^* at the point of the light reception. The expression for the measurable time interval Δt at the point of observation may be found from the simple relations of light propagation. One has

$$t - t_0 = c^{-1}R + \dots , \qquad \boldsymbol{R} = \mathbf{x} - \mathbf{x}_0(t_0) = \mathbf{x} - \mathbf{x}_0^* - \dot{\mathbf{x}}_0^*\Delta t_0 + \dots \tag{8.19}$$

and

$$t^* - t_0^* = c^{-1}R^* + \dots , \qquad \mathbf{R}^* = \mathbf{x}^* - \mathbf{x}_0^* , \qquad \mathbf{x}^* = \mathbf{x}(t^*) \tag{8.20}$$

with

$$R = \rho^* + \mathbf{k}_0\dot{\mathbf{x}}_0^*\Delta t_0 - \mathbf{x}\mathbf{k}_0 + \dots , \qquad R^* = \rho^* - \mathbf{x}^*\mathbf{k}_0 + \dots . \tag{8.21}$$

Taking the difference of (8.19) and (8.20) one gets

$$\Delta t - \Delta t_0 = c^{-1}(R - R^*) + \dots = c^{-1}\mathbf{k}_0(\dot{\mathbf{x}}_0^*\Delta t_0 - \mathbf{x} + \mathbf{x}^*) + \dots \tag{8.22}$$

and finally

$$\Delta t_0 = (1 + c^{-1}\mathbf{k}_0\dot{\mathbf{x}}_0^*)^{-1}(\Delta t + c^{-1}\mathbf{k}_0\mathbf{x} - c^{-1}\mathbf{k}_0\mathbf{x}^*). \tag{8.23}$$

Combination of (8.6), (8.18) and (8.23) enables one to include completely the parallax and proper motion corrections.

9. Geocentric position vectors in BRS and GRS

In addition, it may be reasonable to consider here in more detail as compared with (Bureau des Longitudes, 1997) the relationship between geocentric position vectors in BRS and GRS. Let BRS and GRS metrics be represented, respectively, by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu , \qquad x^0 = ct \tag{9.1}$$

and

$$ds^2 = \hat{g}_{\mu\nu}dw^\mu dw^\nu , \qquad w^0 = cu . \tag{9.2}$$

BRS→GRS transformation (direct transformation) has the form

$$u = t - c^{-2}[A(t) + v_E^k r_E^k] + \dots , \tag{9.3}$$

$$w^i = r_E^i + c^{-2}\Big\{\big[\tfrac{1}{2}v_E^i v_E^k + qF^{ik}(t) + D^{ik}(t)\big]r_E^k + D^{ikm}(t)r_E^k r_E^m\Big\} + \dots \tag{9.4}$$

with

$$r^i_E = x^i - x^i_E(t), \qquad v^i_E = \dot{x}^i_E(t) \tag{9.5}$$

and

$$\dot{A}(t) = \tfrac{1}{2}v^2_E + \bar{U}_E(\mathbf{x}_E). \tag{9.6}$$

GRS→BRS transformation (inverse transformation) has the form

$$t = u + c^{-2}[A(t) + v^k_E w^k] + \dots\,, \tag{9.7}$$

$$x^i = x^i_E(t) + w^i - c^{-2}\Big[\big(\tfrac{1}{2}v^i_E v^k_E + qF^{ik} + D^{ik}\big)w^k + D^{ikm}w^k w^m\Big] + \dots\,. \tag{9.8}$$

Expression (9.8) is not rigorous inverse transformation because its Newtonian right-hand member is not expressed in terms of u (it does not matter for post-Newtonian terms). Let t^* be the BRS moment of time corresponding to event with the GRS coordinates $(u, w^i = 0)$ (Klioner and Voinov, 1993). Then u and t^* are related by the time equation

$$u = t^* - c^{-2}A(t^*) + \dots\,. \tag{9.9}$$

Hence,

$$t - t^* = c^{-2}v^k_E w^k + \dots\,. \tag{9.10}$$

Expanding the first term in the right-hand member of (9.8) in the vicinity of t^* one gets

$$x^i = x^i_E(t^*) + w^i + c^{-2}\Big[\big(\tfrac{1}{2}v^i_E v^k_E - qF^{ik} - D^{ik}\big)w^k - D^{ikm}w^k w^m\Big] + \dots\,. \tag{9.11}$$

By applying the same expansion to the first term of the right-hand member of the direct transformation (9.4) one has

$$w^i = x^i - x^i_E(t^*) + c^{-2}\Big[\big(-\tfrac{1}{2}v^i_E v^k_E + qF^{ik} + D^{ik}\big)r^k_E + D^{ikm}r^k_E r^m_E\Big] + \dots\,. \tag{9.12}$$

Geocentric position vector of some ground station has GRS coordinates w^i taken at some TCG moment u or BRS coordinates $x^i - x^i_E$ taken at moment t^* of TCB time t. Both from (9.11) and (9.12) one gets the same result

$$x^i - x^i_E(t^*) = [1 - c^{-2}\bar{U}_E(\mathbf{x}_E)]w^i + \tfrac{1}{2}c^{-2}(v^k_E w^k)v^i_E - c^{-2}(qF^{ik}w^k + D^{ikm}w^k w^m) + \dots \tag{9.13}$$

or

$$w^i = [1 + c^{-2}\bar{U}_E(\mathbf{x}_E)](x^i - x^i_E(t^*)) - \tfrac{1}{2}c^{-2}(v^k_E r^k_E)v^i_E + c^{-2}(qF^{ik}r^k_E + D^{ikm}r^k_E r^m_E) + \dots\,. \tag{9.14}$$

By introducing functions $z^i_E = z^i_E(u)$

$$z^i_E(u) = x^i_E(t^*) \tag{9.15}$$

one may characterize the motion of the geocenter E in terms of TCB–functions $x^i_E(t)$ in direct transformation (9.3), (9.4) or in terms of TCG–functions $z^i_E(u)$ in inverse transformation (9.7), (9.11). The inverse transformation of such form is used in DSX approach (Damour *et al.*, 1991–1994). The inverse transformation is served, for example, to transform the GRS time–space coordinates (u, w^i) of the terrestrial ground stations into their BRS space–time coordinates (t, x^i). More general relationships for BRS and GRS coordinate differences between moving celestial bodies are given in (Brumberg, 1995b).

10. Conclusion

The main result of the paper consists in the satellite observation reduction formula (4.7) with (8.6), (8.18) and (8.23) relating the measurable light direction $\tilde{p}^{(i)}$ at BRS moment t with the satellite catalog direction $\mathbf{k}_0$ to the light source for the initial epoch t^*_0. It is assumed that the spatial axes of SRS (a satellite catalog) do not rotate in Newtonian sense with respect to global BRS or GRS but to specify the relativistic orientation of the SRS axes one should fix characterizing constants q, $\hat{q}$, $\tilde{q}$. The underlying theory of reference systems and their transformations involved in the derivation of (4.7) is described in (Brumberg, 1995a). The technique of this paper may be combined as well with different solutions of the GRT problem of light propagation including, for example, a solution proposed recently by Kopeikin and Schäfer (1999).

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