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SOME UNCOMPLETED PROBLEMS OF NEWTONIAN AND RELATIVISTIC CELESTIAL MECHANICS

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Some uncompleted problems of Newtonian and relativistic celestial mechanics

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Abstract

. A review of some problems of Newtonian and relativistic celestial mechanics to be regarded as uncompleted problems worthy of further investigation. These problems include general solution of the three-body problem by means of the series of polynomials, construction of the short–term and long–term theories of motion using the fast converging elliptic function expansions, representation of the rotation of the planets in the form compatible with the general planetary theory (combined Birkhoff normalization for the motion and rotation of the planets), determination of the main (indirect) relativistic effects in the motion of a satellite and in the rotation of the primary planet using the Newtonian theories of motion and rotation combined with the relativistic transformation of the reference systems, logical simplification of relativistic celestial mechanics and astrometry in the post–Newtonian approximation by using the linearized (weak–field) metric of general relativity, and the motion of the solar system bodies at the cosmological background.

1. Introduction

This is not an ordinary paper dealing with some specific celestial mechanics problem. The author has worked in the field of celestial mechanics since 1955, the year of graduating from the Moscow state university and entering as a postgraduate to the Institute of Theoretical Astronomy in Leningrad. The period of intensive progress of celestial mechanics started several years later caused by space research, computer technology, advances in mathematics and increase of observational precision. Now this period is practically over. It does not mean that all problems seemed to be actual for that period were completely solved. The aim of the present paper is to outline some problems coming into study by the author but still awaiting and deserving (at least, by the author's opinion) more thorough investigation. These problems were mentioned (amongst some other) as open problems in the draft lectures (Brumberg, 2005) but here they are exposed in more detail.

2. General solution of the three–body problem

In 2012 specialists in celestial mechanics will see the centenary of the famous paper by Sundman (1912) presenting the general solution of the three–body problem by means of the power series converging for any moment of time. More specifically, it was demonstrated by Sundman that for the three–body problem with non–zero value of the area integral (excluding the case of the triple collision) the rectangular coordinates of the bodies, their velocity components and the time are analytic functions in the infinite strip of finite wide seize 2Ω symmetrical with respect to the real axis of the complex plane of variable ω regularizing the double collisions. By applying the transformation by Poincaré of this strip into the unit circle of plane θ

$$\omega = \frac{2\Omega}{\pi}\ln\frac{1+\theta}{1-\theta}, \qquad \theta = \frac{\exp(\pi\omega/2\Omega) - 1}{\exp(\pi\omega/2\Omega) + 1} \tag{2.1}$$

Sundman presented the general three–body solution in form of the series in powers of θ converging for any $|\theta| < 1$, i.e. for any real moment of time t. Not depreciating the theoretical significance of this achievement it was clear from the very beginning that such series are not too useful for gaining an insight into the general picture of motion. But their utility for numerical solution of the three-body problem remained under question. In 1933 Belorizky (1933) gave negative answer to this question demonstrating extremely slow convergence of these series. Since then the Sundman's series have been mentioned in the textbooks on celestial mechanics just as an example of practically useless theoretical result. But even before the research by Sundman it was known that any analytical function $f(\omega)$ represented locally in some circle of convergence by a Taylor series

$$f(\omega) = \sum_{k=0}^{\infty} a_k \omega^k \tag{2.2}$$

can be expanded in its rectilinear star by a uniformly converging series of polynomials whose coefficients are expressed linearly in terms of the coefficients of (2.2) (analytic continuation theorem by Mittag-Leffler). In other words, for any point belonging to the rectilinear star of Mittag-Leffler there exists the sequence of polynomials $f_n(\omega)$ $(n = 1, 2, \ldots)$

$$f_n(\omega) = \sum_{k=0}^{m_n} c_k^{(n)} a_k \omega^k \tag{2.3}$$

converging uniformly to $f(\omega)$. The convergence factors $c_k^{(n)}$ being independent of function $f(\omega)$ represent the coefficients of polynomials

$$g_n(\omega) = \sum_{k=0}^{m_n} c_k^{(n)} \omega^k \tag{2.4}$$

converging uniformly to

$$g(\omega) = \frac{1}{1-\omega} \tag{2.5}$$

provided that ω does not take real values from 1 to ∞. Generally, the faster the sequence (2.4) converges to (2.5), the faster the sequence (2.3) converges to $f(\omega)$. The possibility to present the general solution of the three–body problem by means of the series of polynomials was not mentioned in the paper by Sundman. But this possibility as a consequence of his main theorem was immediately noticed by his contemporaries, first of all by Picard (1913). The first practical realization of this possibility was made only 50 years later (Brumberg, 1963). It turns out that the series of polynomials may be quite plausible for the numerical solution of the three–body problem being concurrent with numerical integration for large intervals of ω. The calculations of (Brumberg, 1963) were performed under very limited computation facilities. Under present computer facilities this problem (a sort of compression of Sundman's series) might be investigated much more efficiently. The paper (Brumberg, 1963) notes also that the general solution of the three–body problem might be represented by means of the series of Hermite polynomials with respect to ω. The domain of convergence of the series of Hermite polynomials is just the strip $|\mathrm{Im}(\omega)| <$ const. In the expansion in Hermite polynomials as well as in the series of polynomials for the rectilinear star of Mittag-Leffler the quantity Ω (demi–width of the strip of convergence) does not occur explicitly (in contrast to series of Sundman) and one may hope for better convergence of these series. Its summation may be facilitated by using the recurrence relations for the Hermite polynomials. Based on this we may formulate

Conjecture 1: The general solution of the three–body problem (triple collisions being excluded) may be efficiently presented in form of series of Hermite polynomials in terms of ω converging for any moment of time.

3. Perturbation theory with elliptic functions

Even when the differential equations of celestial mechanics admit solutions in analytical or semi–analytical form these solutions are often represented by lengthy power–trigonometric series in many variables rather cumbersome in practical use. The effective approach to deal with such series is to apply specialized celestial mechanics routines based on some computer algebra system (such as *Maple, Mathematica*, etc.) in combination with compression techniques of celestial mechanics aimed to make the solutions as compact as possible. Compression of celestial mechanics theories was always a subject of intensive research. More than one century ago Gylden suggested to "view the mean (true, eccentric) anomaly ... as the elliptic amplitude of new (independent) variable" (see Nacozy, 1977). The key point of this idea was not so much to deal with elliptic functions themselves but rather with their fast converging Fourier expansions to be considered as a compression technique in constructing celestial mechanics theories. But the full realization of this idea started only with introducing by E.Brumberg (1992) the elliptic anomaly w defined by

$$\sin g = -\operatorname{cn} u\,, \qquad \cos g = \operatorname{sn} u\,, \qquad w = \frac{\pi u}{2K} - \frac{\pi}{2} \tag{3.1}$$

with eccentric anomaly g, modulus $k = e$ of Jacobi elliptic functions (e being the eccentricity) and the complete elliptic integral of the first kind $K = K(k)$. Theoretical and practical aspects of using the elliptic anomaly were advanced in monographs (Brumberg, 1995; Brumberg and Brumberg, 1999). But the elliptic anomaly is awaiting its actual application. It was too early when in the end of 19th century Gylden put up the idea of the elliptic anomaly in the vague form. It was too late when this idea was clearly formulated in 1992. By that time the rapid advance of celestial mechanics caused by computer facilities and space research demands passed already its top point. In three foregoing decades many theories of motion of natural and artificial celestial bodies were constructed by other techniques available at that time. No doubt that many of these theories could be presented in more compact form by using the elliptic anomaly.

The key point in constructing theories with angular arguments different from time resides in integrating trigonometric functions dependent on several arguments of such type. The classical way was to use the Hansen device to transform two angular arguments into a single one by using the standard expansion with Bessel coefficients

$$\exp \mathrm{i}(A \sin y + B) = \sum_{m=-\infty}^{\infty} J_m(A) \exp \mathrm{i}(my + B), \quad J_{-m}(A) = (-1)^m J_m(A) \tag{3.2}$$

(see details in Brumberg, 1995). But this approach turns out to be not so efficient in practice, especially, in dealing with highly eccentric orbits. Another approach to facilitate the construction of long–term and short–term dynamical theories with using the elliptic anomaly has been proposed in (Brumberg and Brumberg, 2001). The essence of this approach is as follows:

Integration of a series

$$S = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} S_{kl} \exp \mathrm{i}(kx + ly), \qquad S_{00} = 0 \tag{3.3}$$

with respect to time t results in

$$\int S dt = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} S_{kl} I_{kl} \tag{3.4}$$

with integral

$$I_{kl} = \int \exp \mathrm{i}(kx + ly) dt\,. \tag{3.5}$$

Hence, one has to deal with integral (3.5) with k, l being non–zero integers and x, y representing some angular variables (anomalies) relating to one or two bodies. It is assumed that these variables are related to time t by the differential expressions

$$n(1 + f)dt = dx\,, \quad f = f(x), \tag{3.6}$$

and

$$m(1 + \varphi)dt = dy\,, \quad \varphi = \varphi(y) \tag{3.7}$$

where n and m are real constants while f and φ are some functions small by their magnitudes as compared with 1. The case $n = m$ is possible provided that x and y are related to one and the same body. By multiplying (3.6) and (3.7) by k and l, respectively, and adding the results one has

$$(nk + ml)dt = kdx + ldy - (knf + lm\varphi)dt. \tag{3.8}$$

Substituting (3.8) into (3.5) and integrating the first term one gets

$$(nk+ml)I_{kl} = -\,\mathrm{i}\exp\mathrm{i}(kx+ly) - \int (nkf+ml\varphi)\exp\mathrm{i}(kx+ly)\,dt. \tag{3.9}$$

The integral remaining in the right–hand member is often of the same type as the original integral (3.5) but due to the smallness of f and φ it has smaller magnitude than integral (3.5). By repeating this procedure one may hope that at some step of the integration process one may neglect the remaining integral and express the integral (3.5) in the 'closed' form. This procedure may be applied to any number of variables. In classical celestial mechanics a typical example is given with classical anomalies, e.g. $x = v$ (true anomaly) and $y = g$ (eccentric anomaly) with eccentricity e being a small parameter. Then

$$f = \frac{1}{\eta^3}(1+e\cos v)^2 - 1 = \frac{1}{\eta^3}\Big\{1-\eta^3+e\big[\exp\mathrm{i}\,v+\exp(-\,\mathrm{i}\,v)\big]+ \\ +\tfrac{1}{2}e^2+\tfrac{1}{4}e^2\big[\exp 2\,\mathrm{i}\,v+\exp(-2\,\mathrm{i}\,v)\big]\Big\}, \tag{3.10}$$

$$\varphi = \frac{e\cos g}{1-e\cos g} = \frac{e}{2\eta^2}\Big\{\exp\mathrm{i}\,g+\exp(-\,\mathrm{i}\,g)+\tfrac{1}{2}e\big[\exp\mathrm{i}(v-g)+ \\ +\exp\mathrm{i}(-v+g)+\exp\mathrm{i}(v+g)+\exp\mathrm{i}(-v-g)\big]\Big\} \tag{3.11}$$

with $\eta = \sqrt{1-e^2}$. Similar technique has been applied recently to treat the problem of motion of the major planets (Gerasimov *et al.*, 2000). This particular case with v and g may also be treated rigorously by the algorithm of Jefferys (1971).

Extending this classical technique for more sophisticated anomalies let f and φ be represented by the series

$$f = \sum_{r=-\infty}^{\infty} f_r \exp\mathrm{i}\,rx \tag{3.12}$$

and

$$\varphi = \sum_{r=-\infty}^{\infty} \varphi_r \exp\mathrm{i}\,ry\,, \tag{3.13}$$

coefficients f_r and φ_r being, at least, of the order $|r|$ with respect to some small parameter (with vanishing f_0 and φ_0 for zero value of this parameter). By substituting these series into (3.9) one gets a recurrent relation

$$(nk+ml)I_{kl} = -\,\mathrm{i}\exp\mathrm{i}(kx+ly) - \sum_{r=-\infty}^{\infty}(nkf_r I_{k+r,l} + ml\varphi_r I_{k,l+r}). \tag{3.14}$$

Let us assume that coefficients S_{kl} are of the order $|k|+|l|$ with respect to the small parameter and the maximal order of the terms to be taken into account is equal N. It means that we should know coefficients I_{kl} only for $|k|+|l| \le N$ up to the order $N-|k|-|l|$, inclusively. These coefficients may be found by iterations. At first step one gets their initial approximate value from (3.14) by putting $f_r = \varphi_r = 0$. Then the same formula permits one to improve their value until the prescribed accuracy will be achieved. Eventually, the result of the integration (3.4) is presented in the same form as the original series (3.3) itself.

If x is the elliptic anomaly w defined by (3.1) then one of the basic relations (Brumberg and Brumberg, 1999)

$$ndt = (1-k\,\mathrm{sn}\,u)\,\mathrm{dn}\,u\,du \tag{3.15}$$

may be rewritten in form (3.6) with

$$1+f = \frac{\pi}{2K}\frac{1+k\,\mathrm{sn}\,u}{\mathrm{dn}^3\,u}\,. \tag{3.16}$$

The expansion of the right–hand member of (3.16) in multiples of w may be easily found by combining the expansions (6.2.37) and (6.2.39) of (Brumberg and Brumberg, 1999) for $1/dn^3 u$ and $\operatorname{sn} u/\operatorname{dn}^3 u$, respectively, resulting in coefficients f_r of (3.12). As a checking relation one can use

$$(1+f)\Phi = 1, \qquad \Phi \equiv \frac{dM}{dw} = \sum_{s=-\infty}^{\infty} \Phi_s \exp \mathrm{i}\, sw \tag{3.17}$$

where coefficients Φ_s of the derivative of the mean anomaly M with respect to w are given explicitly in (Brumberg and Brumberg, 2001).

This technique may be also applied, for instance, in dealing with two–argument series of the general planetary theory with elliptic functions (Brumberg, 1995). The basic relation (3.4) takes therewith the form

$$\frac{\pi}{2K(k)}(n'-n)(1+f)dt = dw \tag{3.18}$$

with n, n' being the mean motions of the disturbed and disturbing planets, respectively, modulus k of elliptic functions being expressed in terms of semi–major axes a and a'

$$k^2 = \frac{4aa'}{(a+a')^2} \tag{3.19}$$

and $K(k)$ being the complete elliptic integral of the first kind. Then,

$$f = \frac{1}{\operatorname{dn} u} - 1\,, \qquad u = \frac{K}{\pi}\, w\,. \tag{3.20}$$

The standard trigonometric expansion of $1/\operatorname{dn} u$ results in (3.10) with $x = w$ and coefficients

$$f_0 = \frac{\pi}{2k'K} - 1\,, \qquad f_{-r} = f_r = \frac{(-1)^{|r|}\pi}{k'K}\,\frac{q^{|r|}}{1+q^{2|r|}}\,, \tag{3.21}$$

k' and q being the complementary modulus and Jacobi nome, respectively.

In the second–order general planetary theory one deals with the triplet of planets i (disturbed planet), and j and k (disturbing planets). Introducing anomalies w_{ij} and w_{ik} by analogy with (3.18)–(3.20) and putting $x = w_{ij}$, $y = w_{ik}$ one may apply the technique described above to integrate the two-argument series of the right–hand members. This technique might be more effective than the Hansen device to express one of the anomalies in terms of another one (Brumberg, 1995). In addition, this integration technique may be applied to any angular variables satisfying relations (3.6) and (3.12) including various regularizing arguments (KS and others) for many–body problem. But only practical applications may demonstrate actual efficiency of this technique. Therefore, it is reasonable to formulate

Conjecture 2: Integration algorithm (3.3)–(3.7), (3.12)–(3.14) might be effective tool in constructing semi–analytical theories of motion, particularly, with using the elliptic anomaly (3.1) (for highly–eccentric orbits) or synodic anomaly (3.18)–(3.20) (for general planetary theory).

4. General theory of planetary motion and rotation

In no way underestimating numerical integration planetary theories (DE of JPL, EPM of IAA) or Le Verrier type semi–analytical theories (VSOP of IMCCE) the construction of GPT (general planetary theory) still presents a challenge for celestial mechanics (Brumberg, 1995). The aim of GPT is to separate short–period (dependent on mean longitudes) and long–period (dependent on perihelia and nodes longitudes) terms reducing the equations of motion to a secular system describing long–term planetary evolution. This system may be solved by various techniques in different forms (including theoretically attractive but practically hardly feasible trigonometric form) but this specificity is not demanded by GPT. GPT has been elaborated at first for eight major planets (Brumberg, 1995 and references therein) and then was extended to include the Moon (Brumberg and Ivanova, 1985). If M_i and $\mathbf{r}_i$ are masses and heliocentric position vectors of the major planets for $i = 1, 2, \ldots, 8$ with the value $i = 3$ relating to the Earth–Moon barycentre, and M_9 and $\mathbf{r}_9$ are the mass of the Moon and its geocentric position vector, respectively, then designating the rectangular coordinates by x, y, z we have for $i = 1, 2, \ldots, 9$

$$x_i + \mathrm{i}\, y_i = A_i(1 - p_i)\zeta_i\,, \quad z_i = A_i w_i\,, \quad \zeta_i = \exp \mathrm{i}\, \lambda_i\,, \quad \lambda_i = n_i t + \varepsilon_i \tag{4.1}$$

with

$$n_i^2 A_i^3 = G(M_0 + M_i) \quad (i = 1, 2, \ldots, 8), \qquad n_9^2 A_9^3 = G M_3\,, \tag{4.2}$$

index 0 referred to the Sun. In GPT one takes, as a starting point, a quasi–periodic intermediary with arbitrary constants n_i and ε_i and then constructs a normalizing transformation from p_i, q_i, w_i to eccentric and oblique variables $a_i, \bar{a}_i, b_i, \bar{b}_i$ admitting the immediate reduction to the secular system. The resulting equations have the form

$$\dot{X} = \mathrm{i}\, \mathcal{N}\big[PX + R(X, t)\big]. \tag{4.3}$$

Here $X = (a, \bar{a}, b, \bar{b})$ and $R = (R_1, \ldots, R_4)$ are vectors with 36 components (a, b and R_i are 9–vectors). One has therewith $R_2 = -\bar{R}_1$, $R_4 = -\bar{R}_3$. $\mathcal{N}$ and P are 36×36 diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(N, N, N, N), \qquad P = \mathrm{diag}(E, -E, E, -E), \tag{4.4}$$

N is 9×9 diagonal matrix of mean motions n_i and E is 9×9 unitary matrix. The GPT right–hand members are

$$R_1 = DA\bar{D}a + D\Phi\,, \quad R_2 = -\bar{R}_1\,, \quad R_3 = DB\bar{D}b + D\Psi\,, \quad R_4 = -\bar{R}_3 \tag{4.5}$$

with 9×9 diagonal matrix $D = \mathrm{diag}(\exp \mathrm{i}\, \lambda_k)$. The transformation

$$a = D\alpha\,, \qquad b = D\beta \tag{4.6}$$

results in the autonomous secular system for the major planets and the Moon with constant matrices A, B in the linear parts and the power series $\Phi = \Phi(\alpha, \bar{\alpha}, \beta, \bar{\beta})$, $\Psi = \Psi(\alpha, \bar{\alpha}, \beta, \bar{\beta})$ in the non–linear parts

$$\dot{\alpha} = \mathrm{i}\, N[A\alpha + \Phi(\alpha, \bar{\alpha}, \beta, \bar{\beta})], \qquad \dot{\beta} = \mathrm{i}\, N[B\beta + \Psi(\alpha, \bar{\alpha}, \beta, \bar{\beta})]. \tag{4.7}$$

Actual construction of the GPT to higher degrees with respect to eccentric and oblique variables and higher order with respect to the ratio of the planetary masses to the solar mass seems now to be quite feasible especially with using synodic anomalies related with elliptic function expansions as indicated in Conjecture 2.

The next logical step is to try to study the rotation of celestial bodies within the GPT framework, i.e. to separate short–term and long–term rotation terms and to obtain a secular system for the long–term rotation compatible with the secular system for planetary motion. The investigation of this problem started, for simplicity, with Poisson equations for the case of the axially symmetrical Earth gave promising results (Brumberg and Ivanova, 1997, 2007). The general case of the rigid–body rotation of the Earth as well as of any other solar system body may be hopefully treated in the same manner. The important starting point here is to present the equations of rotation in the form compatible with GPT equations (4.3). Initially, in the Earth's rotation problem one has to deal with two geocentric coordinate systems, an inertial system $\mathbf{x} = (x_i)$ with the fixed ecliptic as a main reference plane and a rotating, Earth–fixed system $\mathbf{x}' = (x_i')$ oriented by means of the principal axes of inertia. The rotation matrix Λ of the transformation

$$\mathbf{x} = \Lambda \mathbf{x}'\,, \qquad \mathbf{x}' = \Lambda^T \mathbf{x} \tag{4.8}$$

can be expressed in terms of Euler angles ψ, θ, φ (Tisserand, 1891; Smart, 1953)

$$\Lambda = D_3(\psi) D_1(\theta) D_3(-\varphi)\,, \tag{4.9}$$

D_i being elementary rotation matrices (e.g., Brumberg, 1995). To solve the Earth's rotation problem means to find these Euler angles in function of time. For computer manipulation it is reasonable to replace three Euler angles by four Euler parameters regarded as the components of the unitary 4–vector $\mathbf{u} = (u_1, u_2, u_3, u_4)$. With the aid of the complex variables $u = u_1 + \mathrm{i}\, u_2$, $v = u_3 + \mathrm{i}\, u_4$ there results

$$u = -\sin\frac{\theta}{2}\exp\left(-\mathrm{i}\,\frac{\psi+\varphi}{2}\right), \qquad v = \mathrm{i}\cos\frac{\theta}{2}\exp\left(\mathrm{i}\,\frac{\psi-\varphi}{2}\right). \tag{4.10}$$

In terms of Euler parameters the kinematical and dynamical equations of the Earth's rotation are described in the form

$$2\dot{\mathbf{u}} = \Omega(\boldsymbol{\omega})\mathbf{u}\,, \qquad I_i\dot{\omega}_i - \sum_{j,k=1}^{3} \varepsilon_{ijk} I_j \omega_j \omega_k = M_i \quad (i = 1,2,3) \tag{4.11}$$

with principal inertia moments I_i ($I_1 \le I_2 < I_3$ for the Earth) and vector $\boldsymbol{\omega} = (\omega_i)$ of the Earth's rotation angular velocity with components ω_i referred to x_i' axes. Ω is 4×4 matrix with components

$$\Omega_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk}\omega_k\,, \quad \Omega_{i4} = -\Omega_{4i} = \omega_i\,, \quad \Omega_{44} = 0. \tag{4.12}$$

In (4.11) and (4.12) ε_{ijk} stands for the Levi-Civita symbol

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i).$$

The torque vector $\mathbf{M} = (M_i)$ is determined by

$$2\mathbf{M} = Q(\mathbf{u})\mathrm{grad}_u U \tag{4.13}$$

with the force function U and 4×4 matrix Q with components

$$Q_{ij} = \sum_{k=1}^{3} \varepsilon_{ijk} u_k + \delta_{ij} u_4\,, \quad Q_{4i} = -Q_{i4} = u_i\,, \quad Q_{44} = u_4\,, \tag{4.14}$$

δ_{ij} being the Kronecker symbol. In application to the rigid Earth model it is reasonable to put (under the notation of the Connaissance des Temps)

$$\omega_1 = \Omega m_1\,, \qquad \omega_2 = \Omega m_2\,, \qquad \omega_3 = \Omega(1+m_3)\,, \tag{4.15}$$

Ω being the mean Earth's rotation velocity ($\Omega = 7.292115 \cdot 10^{-5}$ rad/s). The dimensionless quantities m_i are small ($m_1 \sim m_2 \sim 10^{-6}$, $m_3 \sim 10^{-8}$). To be consistent with our previous designations we put $\Omega = -2n$, $n =$ const. The equations of the Earth's rotation may be put into the form

$$\dot{u} = \mathrm{i}\, n[(1+m_3)u - m\bar{v}], \tag{4.16}$$

$$\dot{v} = \mathrm{i}\, n[(1+m_3)v + m\bar{u}], \tag{4.17}$$

$$\dot{m} = -2\,\mathrm{i}\, n(1+m_3)[(k_1+k_2)m + (k_1-k_2)\bar{m}] - \frac{1}{2n}\left(\frac{M_1}{I_1} + \mathrm{i}\,\frac{M_2}{I_2}\right), \tag{4.18}$$

$$\dot{m}_3 = -\mathrm{i}\, n\frac{I_2 - I_1}{2I_3}(m^2 - \bar{m}^2) - \frac{M_3}{2nI_3} \tag{4.19}$$

with

$$m = m_1 + \mathrm{i}\, m_2\,. \tag{4.20}$$

and

$$k_1 = \frac{I_3 - I_1}{2I_2}\,, \qquad k_2 = \frac{I_3 - I_2}{2I_1}\,. \tag{4.21}$$

Equation (4.18) may be simplified by means of the change of the variables

$$m_1 = 2\sqrt{k_2}\, m_1' , \qquad m_2 = 2\sqrt{k_1}\, m_2' , \qquad m' = m_1' + \mathrm{i}\, m_2' \tag{4.22}$$

involving

$$\dot m' = -4\,\mathrm{i}\, n\sqrt{k_1 k_2}\,(1+m_3)m' - \frac{1}{4n}\left(\frac{1}{\sqrt{k_2}}\frac{M_1}{I_1} + \frac{\mathrm{i}}{\sqrt{k_1}}\frac{M_2}{I_2}\right). \tag{4.23}$$

In using (4.23) instead of (4.18) one should use in the right–hand members of equations (4.16), (4,17) and (4.19) the expressions

$$m = \left(\sqrt{k_2}+\sqrt{k_1}\right) m' + \left(\sqrt{k_2}-\sqrt{k_1}\right)\bar m', \tag{4.24}$$

and

$$m^2 - \bar m^2 = 4\sqrt{k_1 k_2}\,(m'^2 - \bar m'^2). \tag{4.25}$$

We have derived the system of the equations of the first order in terms of u, $\bar u$, v, $\bar v$, m, $\bar m$, m_3 (with possible changing m, $\bar m$ by m', $\bar m'$). By differentiating (4.16), (4.17) we can exclude m and m_3 to deal with the system of the second order equations in terms of u , $\bar u$, v , $\bar v$ alone. There results

$$\ddot u = -n^2(u+Q_1), \tag{4.26}$$

$$\ddot v = -n^2(v+Q_2), \tag{4.27}$$

with

$$Q_1 = (2m_3 + m_3^2 + m\bar m)u - \frac{\mathrm{i}}{n}(\dot m_3 u - \dot m \bar v), \tag{4.28}$$

$$Q_2 = (2m_3 + m_3^2 + m\bar m)v - \frac{\mathrm{i}}{n}(\dot m_3 v + \dot m \bar u), \tag{4.29}$$

where $\dot m$ and $\dot m_3$ are expressed by (4.18), (4.19) with

$$m = \frac{\mathrm{i}}{n}(\dot u v - \dot v u) \tag{4.30}$$

and

$$m_3 = -1 - \frac{\mathrm{i}}{n}(\dot u \bar u + \dot v \bar v). \tag{4.31}$$

Equations (4.26), (4.27) (together with their conjugate complements) represent a system of four second–order differential equations with respect to u, $\bar u$, v, $\bar v$. To facilitate the application of the Birkhoff normalization we transform this system to a special–form system of eight differential equations of the first order with respect to new variables $p = (p_i)$, $\bar p = (\bar p_i)$ $(i = 1, \ldots, 4)$. Indeed, the transformation

$$p_1 = u - \frac{\mathrm{i}}{n}\,\dot u\,, \quad p_2 = \bar u - \frac{\mathrm{i}}{n}\,\dot{\bar u}\,, \quad p_3 = v - \frac{\mathrm{i}}{n}\,\dot v\,, \quad p_4 = \bar v - \frac{\mathrm{i}}{n}\,\dot{\bar v} \tag{4.32}$$

brings the system (4.26), (4.27) to the system like (4.3)

$$\dot X = \mathrm{i}\,\mathcal{N}[PX + R(X,t)] \tag{4.33}$$

where X and R stand for 8–vectors of new variables and right–hand members, respectively,

$$X = (p, \bar p), \qquad R = (Q\,, -\bar Q), \qquad Q = (Q_1\,, \bar Q_1\,, Q_2\,, \bar Q_2), \tag{4.34}$$

$\mathcal{N}$ and P are 8×8 diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(n,n,n,n,n,n,n,n), \qquad P = \mathrm{diag}(1,\,1,\,1,\,1,-1,-1,-1,-1). \tag{4.35}$$

The right–hand members of the equations of the Earth's rotation depend on planetary and lunar coordinates. Usually, these coordinates are regarded as known functions of time. Instead of this, they may be regarded them as functions satisfying some definite differential equations. Combining (4.3) and (4.33) one obtains a complete system describing the planetary and lunar motions and Earth's rotation. The resulting equations have the same form as (4.3) or (4.33) where

$$X = (a, \bar{a}, b, \bar{b}, p, \bar{p}), \qquad R = (R_1, \ldots, R_6) \tag{4.36}$$

are vectors with 44 components (a, b and R_i for $i = 1, 2, 3, 4$ are 9–vectors, p and $R_5 = Q$ are 4–vectors). One has therewith $R_2 = -\bar{R}_1$, $R_4 = -\bar{R}_3$, $R_6 = -\bar{R}_5$. $\mathcal{N}$ and P are 44×44 diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(N, N, N, N, n, n, n, n, n, n, n, n, n),$$

$$P = \mathrm{diag}(E_{(9)}, -E_{(9)}, E_{(9)}, -E_{(9)}, E_{(4)}, -E_{(4)}), \tag{4.37}$$

N is 9×9 diagonal matrix of mean motions n_i, $E_{(9)}$ and $E_{(4)}$ are unitary matrices of dimension 9×9 and 4×4, respectively. This form allows us to formulate

Conjecture 3: It is possible to extend GPT to treat not only planetary–lunar motions but their own rotations as well and to obtain a unique secular system describing evolution of motion and rotation.

5. Relativistic extension of Newtonian theories of motion and rotation

With a few exceptions, general relativity is used now in practical celestial mechanics mostly in representing the motion of the major planets in BRS (barycentric reference system). No doubt that rather soon new relativistic theories of the motion of the satellites of the major planets as well as of the rotation of the planets will be highly demanded. A question arises if in so doing it is possible to use the existing Newtonian semi–analytical theories of motion and rotation. In (Brumberg, 2004) dealing with motion of an Earth satellite in GRS (geocentric reference system) it was suggested for the evaluation of the third–body perturbations to use not the rigorous (rather cumbersome) relativistic equations of motion but rather the Newtonian equations (with relativistic contributions from the primary) with substitution of the relativistic four–dimensional [BRS↔GRS] transformation (e.g., Bretagnon and Brumberg, 2003) for the coordinates of the perturbing bodies (the Sun, the Moon). It simplifies the computation of the main (indirect) relativistic perturbations in the satellite motion caused by the influence of the third–body. Brumberg and Simon (2003, 2007) demonstrated the possibility to find the indirect relativistic perturbations in the Earth's rotation just by substituting the [BRS↔GRS] transformation for the lunar–solar GRS coordinates into the existing Newtonian theory (SMART97) of the Earth's rotation. The determination of the relativistic indirect terms in the Earth's rotation parameters opens way to find the relativistic contributions in the rotation vector of GRS→ITRS (International Terrestrial Reference System) transformation as given in (Bretagnon and Brumberg, 2003).

This technique is exposed in detail in the papers indicated above. For convenience, the basic formulas underlying this technique are reproduced below (within the post–Newtonian approximation).

Let t and $\mathbf{x} = (x^i)$ ($i = 1, 2, 3$) denote BRS time and spatial coordinates. The corresponding variables for GRS let be denoted by u and $\mathbf{w} = (w^i)$. The relationship between t and u reads

$$u = t - c^{-2}[A(t) + \mathbf{v}_E \mathbf{r}_E] + \ldots, \qquad \mathbf{r}_E = \mathbf{x} - \mathbf{x}_E\,, \tag{5.1}$$

$\mathbf{r}_E$, $\mathbf{v}_E$ being the BRS position vector and velocity of the Earth. The time function $A(t)$ satisfies the equation

$$\dot{A}(t) = \tfrac{1}{2}\mathbf{v}_E^2 + \bar{U}_E(t, \mathbf{x}_E), \qquad \bar{U}_E(t, \mathbf{x}_E) = \sum_{A \neq E} \frac{GM_A}{r_{AE}} \tag{5.2}$$

with summation over all perturbing bodies A taken into account (the Sun and the Moon in the cases considered above) and $\mathbf{r}_{AE} = \mathbf{x}_A - \mathbf{x}_E$. Its solution is presented with separating a linear secular term from all other (polynomial, trigonometric and mixed) terms

$$A(t) = c^2 L_C t + A_p(t)\,, \qquad L_C t = L_C (J - 2443144.5) 86400 s \tag{5.3}$$

with

$$c^{-2}A_p(t) = P = \sum_\alpha t^\alpha \left[\sum_k A_k^\alpha \cos(\psi_k^\alpha + \nu_k^\alpha t)\right] \tag{5.4}$$

and condition $P = 0$ on Jan. 1, 1977 0h 0m 0s TAI (J=2443144.5 TAI). Theoretically, t and u are supposed to be TCB and TCG, coordinate time scales of BRS and GRS, respectively. But in practice t and u are often used as the time scales TDB and TT differing by scalar factors from TCB and TCG, respectively,

$$\mathrm{TDB} = (1 - L_B)\mathrm{TCB}\,, \qquad \mathrm{TT} = (1 - L_G)\mathrm{TCG}\,. \tag{5.5}$$

These three scalar factors satisfy the relation

$$1 - L_B = (1 - L_C)(1 - L_G), \qquad (L_B = L_C + L_G - L_C L_G). \tag{5.6}$$

According IAU Resolution B1 (2000) their values read

$$L_C = 1.48082686741 \times 10^{-8}\,, \qquad L_B = 1.55051976772 \times 10^{-8}\,, \tag{5.7}$$

$$L_G = 6.969290134 \times 10^{-10}\,. \tag{5.8}$$

Contrary to values (5.7) dependent on the O–C analysis of planetary–lunar motion L_G is a defining constant. In terms of TDB and TT the equation (5.1) takes the form

$$\mathrm{TT} = \mathrm{TDB} - c^{-2}[A_p(t) + \mathbf{v}_E\mathbf{r}_E] + \ldots\,. \tag{5.9}$$

The use of TDB and TT involves the scale factors for spatial coordinates and mass coefficients

$$(\mathbf{x})_{\mathrm{TDB}} = (1 - L_B)\mathbf{x}\,, \qquad (GM)_{\mathrm{TDB}} = (1 - L_B)GM\,, \tag{5.10}$$

$$(\mathbf{w})_{\mathrm{TT}} = (1 - L_G)\mathbf{w}\,, \qquad (G\hat{M})_{\mathrm{TT}} = (1 - L_G)(G\hat{M})\,, \tag{5.11}$$

so that the velocity components and the equations of motion remain the same. By introducing the scalar parameters

$$\mu = \begin{cases} 1\,, & t = \mathrm{TCB}\,, \\ 0\,, & t = \mathrm{TDB}\,, \end{cases} \qquad \nu = \begin{cases} 1\,, & u = \mathrm{TCG}\,, \\ 0\,, & u = \mathrm{TT} \end{cases} \tag{5.12}$$

one may write the direct BRS↔GRS transformation for any possible combination of the time scales as follows:

$$u = (1 - \mu L_B + \nu L_G)t - c^{-2}(A_p + \mathbf{v}_E\mathbf{r}_E), \tag{5.13}$$

$$w^i = [1 + (1-\mu)L_B - (1-\nu)L_G]r_E^i + c^{-2}\Lambda^i(t,\,\mathbf{r}_E), \tag{5.14}$$

$$\Lambda^i(t,\,\mathbf{r}_E) = \tfrac{1}{2}\mathbf{v}_E\mathbf{r}_E v_E^i - q\varepsilon_{ijk}F^j r_E^k + \bar{U}_E(t,\mathbf{x}_E)r_E^i + \mathbf{a}_E\mathbf{r}_E r_E^i - \tfrac{1}{2}\mathbf{r}_E^2 a_E^i\,, \tag{5.15}$$

where $\mathbf{a}_E$ is the BRS acceleration of the Earth, F^j is the vector of geodesic rotation and q is a numerical parameter to distinguish between kinematically non–rotating ($q = 0$) or dynamically non–rotating ($q = 1$) GRS (e.g., Bretagnon and Brumberg, 2003). Inverse transformation reads

$$t = (1 + \mu L_B - \nu L_G)u + c^{-2}(A_p + \mathbf{v}_E\mathbf{w}), \tag{5.16}$$

$$x^i = [1 - (1-\mu)L_B + (1-\nu)L_G](w^i + z_E^i) + c^{-2}\Gamma^i(u,\,\mathbf{w}), \tag{5.17}$$

$$\Gamma^i(u,\,\mathbf{w}) = \tfrac{1}{2}\mathbf{v}_E\mathbf{w}v_E^i + q\varepsilon_{ijk}F^j w^k - \bar{U}_E(t,\mathbf{x}_E)w^i - \mathbf{a}_E\mathbf{w}w^i + \tfrac{1}{2}\mathbf{w}^2 a_E^i\,. \tag{5.18}$$

Function $z_E^i = z_E^i(u)$ representing the Earth's motion referred to the GRS coordinate time can be computed from the BRS–time representation of the Earth's motion by means of

$$z_E^i(u) = [1 + (1-\mu)L_B - (1-\nu)L_G]x_E^i(t^*), \tag{5.19}$$

where

$$t^* = (1 + \mu L_B - \nu L_G)u + c^{-2}A_p\,. \tag{5.20}$$

Instead of determining t^* one can just use

$$x^i_E(t^*) = x^i_E[(1 + \mu L_B - \nu L_G)u] + c^{-2}A_p v^i_E\,. \tag{5.21}$$

By substituting (5.19) into (5.17) one can use also

$$x^i = [1 - (1-\mu)L_B + (1-\nu)L_G]w^i + x^i_E(t^*) + c^{-2}\Gamma^i(u, \mathbf{w}). \tag{5.22}$$

The right–hand members of the Newtonian equations of the Earth's satellite motion and the Earth's rotation involve the geocentric position vectors of the disturbing bodies $w^i_A(u)$. The key point of the technique of this section is to substitute their expressions resulted from the BRS↔GRS transformation as follows:

$$w^i_A(u) = z^i_A(u) - z^i_E(u) + c^{-2}\big[\Lambda^i(t^*, \mathbf{r}_{AE}) + \mathbf{v}_E\mathbf{r}_{AE}v^i_{AE}\big], \tag{5.23}$$

$$z^i_A(u) - z^i_E(u) = [1 + (1-\mu)L_B - (1-\nu)L_G][x^i_A(t^*) - x^i_E(t^*)] \tag{5.24}$$

with $\mathbf{v}_{AE} = \mathbf{v}_A - \mathbf{v}_E$. These expressions contain relativistic terms leading to the indirect relativistic perturbations in the right–hand members of the GRS equations of motion and rotation. The extension of this technique for other planets of the solar system is straightforward.

This technique facilitates greatly the determination of the main relativistic perturbations in the primary planet coordinate system. To investigate and to apply this technique more widely it seems reasonable to formulate

Conjecture 4: The main (indirect) relativistic perturbations in the geocentric motion of an Earth satellite (including the Moon) and in the rotation of the Earth may be obtained by substituting into the Newonian GRS equations the relativistic four–dimensional BRS↔ GRS transformation for the coordinates of the perturbing bodies (with extension of this technique for any planet of the solar system).

6. Is the linearized (weak–field) GRT metric sufficient for relativistic celestial mechanics and astrometry?

Nowadays, thanks to the PNA (post–Newtonian approximations) techniques of GRT (general relativity theory) one has in practical disposition the expansions of the metric coefficients in quasi–Galilean coordinates

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu\,, \qquad x^0 = ct\,, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \tag{6.1}$$

$$\eta_{00} = 1, \qquad \eta_{0i} = 0, \qquad \eta_{ij} = -\delta_{ij} \tag{6.2}$$

for the gravitational N–body field as follows:

$$h_{00} = c^{-2}\underset{2}{h_{00}} + c^{-4}\underset{4}{h_{00}} + c^{-5}\underset{5}{A_{00}} + c^{-6}\underset{6}{h_{00}} + c^{-7}\underset{7}{A_{00}} + \mathrm{O}(c^{-8})\,, \tag{6.3}$$

$$h_{0i} = c^{-3}\underset{3}{h_{0i}} + c^{-5}\underset{5}{h_{0i}} + c^{-6}\underset{6}{A_{0i}} + \mathrm{O}(c^{-7})\,, \tag{6.4}$$

$$h_{ij} = c^{-2}\underset{2}{h_{ij}} + c^{-4}\underset{4}{h_{ij}} + c^{-5}\underset{5}{A_{ij}} + \mathrm{O}(c^{-6}) \tag{6.5}$$

(Greek indices running from 0 to 3, Latin indices running from 1 to 3, c being the light velocity in vacuum). The $A_{\mu\nu}$ terms are due to gravitational radiation of the N–body system presenting a qualitative difference from the Newtonian N–body problem (see references in Brumberg, 1991). Such advanced expansion is needed in studying the motion in a strong gravitational field (e.g., the binary pulsar problem). In present applications related to the solar system much more restricted approximations are needed. With all $h_{\mu\nu} = 0$ metric (6.1) becomes Galilean (Minkowski) metric of SRT (special relativity theory) yielding the Newtonian equations of light propagation (motion in a straight line). Retaining in h_{00} only the first term $\underset{2}{h}_{00}$ and rejecting all $h_{0i}\,, h_{ij}$ one gets the metric giving the Newtonian equations of motion and incorrect approximation for the light propagation equations. The correct post–Newtonian equations of light propagation are obtained by the metric with $\underset{2}{h}_{00}\,,\ \underset{3}{h}_{0i}$ and $\underset{2}{h}_{ij}$. This metric is called linearized (weak–field) GRT metric and it is quite sufficient for present relativistic astrometry based on the post–Newtonian equations of light propagation and post–Newtonian (generalized Lorentz) transformations between different four–dimensional reference systems (including the post–Newtonian theory of the time scales).

It is widely believed that to derive the post–Newtonian equations of material bodies underlying present relativistic celestial mechanics it is necessary to include in h_{00} also the non–linear fourth–order term $\underset{4}{h}_{00}$. Indeed, the post–Newtonian equations of motion of a test particle in a given gravitational field resulting from the geodesic principle (motion on a geodesic line) demand $\underset{4}{h}_{00}$ (e.g., Section 2.2.3 of Brumberg, 1991). The derivation of the equations of motion in the post–Newtonian N–body problem by the PNA techniques also demands this term.

Nevertheless, it may be reminded that as far back as 1957 Infeld (1957) suggested to derive the post–Newtonian equations of the N–body problem from the variational principle for the GRT field equations. This technique turned out to be the most economical one not demanding the use of $\underset{4}{h}_{00}$. In doing so Infeld used the mass tensor with δ– functions. In (Brumberg, 1972) the same technique was applied in combination with the Fock mass tensor for ideal liquid. The possibility to use just the GRT linearized metric for relativistic celestial mechanics simplifies greatly the use of GRT in practical astronomy. Based on the variational principle all practical tools of relativistic celestial mechanics such as EIH (Einstein–Infeld–Hoffman) equations of N–body problem, equations of motion of a test particle in a given field of gravitating masses, equations of rotational motion of celestial bodies, etc., may be derived just from the linearized metric. Very regretfully, this idea was forgotten (rather frequent case in any science in all times). This idea was reminded in (Brumberg, 1997) indicating that the IAU Resolutions of 1991 fixing $\underset{2}{h}_{00}$ and $\underset{2}{h}_{ij}$ should be extended by adding $\underset{3}{h}_{0i}$ (and not $\underset{4}{h}_{00}$). But both $\underset{3}{h}_{0i}$ and $\underset{4}{h}_{00}$ terms were explicitly indicated in the IAU (2000) Resolution B1 (IAU, 2001) on reference systems and time scales. Moreover, $\underset{4}{h}_{00}$ term is combined with $\underset{2}{h}_{00}$ term by introducing generalized (relativistic) potential instead of 'out-of-date' Newtonian potential.

Trying to revive the idea by Infeld and to show its practical prospects the author presented to *Celestial Mechanics and Dynamical Astronomy* journal a paper under the title of this section. This title seemed to be too polemic to the reviewers and has been replaced by a more moderate title (Brumberg, 2007). In what follows we reproduce the basic formulations of this paper.

The starting point is the GRT field equations

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = -\kappa T^{\mu\nu} \qquad (\kappa = 8\pi G c^{-2}) \tag{6.6}$$

expressed in terms of the Ricci tensor $R^{\mu\nu}$, scalar curvature R, metric tensor $g^{\mu\nu}$ and mass tensor

$$c^2 T^{\mu\nu} = (c^2\rho^* + \rho^*\Pi + p) u^\mu u^\nu - p g^{\mu\nu}\,. \tag{6.7}$$

The representation (6.7) is due to Fock (1955) with the invariable rest mass density ρ^* satisfying the equation of continuity, potential compressional energy Π, pressure p and 4–velocity u^μ

$$u^\mu = \frac{dx^\mu}{ds}\,, \qquad \sqrt{-g}\,\rho^* = \rho\frac{ds}{dx^0}\,, \qquad d\Pi = \frac{p}{{\rho^*}^2} d\rho^*\,, \tag{6.8}$$

ρ being the rest mass density.

The field equations (6.6) may be derived from the condition of stationarity of some scalar invariant. This variational principle may be presented in the form

$$\delta \int \Big[\frac{c^4}{16\pi G}\sqrt{-g}J + c^2(1 + c^{-2}\Pi)\rho\frac{ds}{dx^0}\Big] d\Omega = 0 \tag{6.9}$$

with changing $g_{\mu\nu}$ provided that the variations of $g_{\mu\nu}$ and their first derivatives vanish on the boundary of the 4–domain of integration $d\Omega$. Here J denotes

$$J = g^{\mu\nu}(\Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\alpha\mu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}), \tag{6.10}$$

$\Gamma^{\alpha}_{\mu\nu}$ being the Christoffel symbols (Fock, 1955; Brumberg, 1972) Here and everywhere below the partial derivative with respect to some coordinate is denoted by a comma accompanied by the appropriate index. Einstein summation rule is applied everywhere assuming the summation over any twice repeated Greek or Latin index. It may be noted that J as a function of $g^{\mu\nu}$ and derivatives $\partial g^{\mu\nu}/\partial x^{\alpha}$ is Lagrangian for the left–side of the field equations (6.6).

Since the elementary 4–volume is $d\Omega = cdt\, d^3x$ it seems reasonable to rewrite this principle

$$\delta \int L dt = 0 \tag{6.11}$$

to treat L as the Lagrangian of the N–body problem equations (see Infeld and Plebansky, 1960 for mathematical reasoning). Needless to say, this may be done in the post–Newtonian and post–post–Newtonian approximations in (6.3)–(6.5) but not for the radiation approximation when the equations of motion cannot be presented in the Lagrange form due to the presence of the radiative (dissipative) terms. Changing the sign one gets from (6.9) and (6.11)

$$L = -\int \Big[\frac{c^4}{16\pi G}\sqrt{-g}J + c^2(1 + c^{-2}\Pi)\rho\frac{ds}{dx^0}\Big] d^3x\,, \tag{6.12}$$

the integral is to be considered as the sum of integrals taken over the volumes of the bodies.

One may note that due to the geodesic principle the derivative ds/dx^0 leads to the Lagrangian $c^2(1-ds/dx^0)$ of the geodesic equations of a test particle in a given field. It will be seen below that both parts of the integrand in (6.12) contain explicitly $\underset{4}{h}_{00}$ but with opposite signs resulting in cancelling this term in the integrand as a whole.

The well–known general solution of the linearized field equations (the linearized metric) in arbitrary quasi–Galilean coordinates reads (e.g., Brumberg, 1991)

$$\begin{aligned} ds^2 =& (1 - c^{-2}2U)\, c^2dt^2 + 2(c^{-3}4U^i + a_{0,i} + a_{i,0})\, cdt\, dx^i + \\ &+ [-(1 + c^{-2}2U)\delta_{ij} + a_{i,j} + a_{j,i}]\, dx^i dx^j \end{aligned} \tag{6.13}$$

with Newtonian potential U and vector–potential U^i satisfying the equations

$$U_{,ss} = -4\pi G\rho\,, \qquad U^i_{,ss} = -4\pi G\rho v^i \tag{6.14}$$

with integral representations

$$U = G\int \frac{\rho'}{|\,\mathbf{r} - \mathbf{r}'|}\, d^3x'\,, \qquad U^i = G\int \frac{\rho' v'^i}{|\,\mathbf{r} - \mathbf{r}'|}\, d^3x'\,. \tag{6.15}$$

One may add to it the relation

$$cU_{,0} + U^i_{,i} = 0\,. \tag{6.16}$$

In (6.13) a_μ are four arbitrary functions of x^0, x^1, x^2, x^3 (vanishing for the case of harmonic coordinates). In dealing with these functions a_0 is to be regarded as a third–order function and a_i as second–order functions. The coordinate conditions with non–zero a_i are used now very seldom. But together with harmonic choice $a_0 = 0$ one may often meet the so called SPN (standard post-Newtonian) gauge with

$$a_i = 0\,, \qquad a_0 = c^{-2}\nu\chi_{,0}\,, \qquad \nu = \begin{cases} 0\,, & \text{harmonic gauge} \\ 1\,, & \text{SPN gauge} \end{cases} \tag{6.17}$$

and

$$\chi_{,ss} = U\,, \qquad \chi = \frac{1}{2}G\int \rho'\,|\,\mathbf{r} - \mathbf{r}'|\, d^3x' \tag{6.18}$$

resulting to

$$\underset{3}{h}_{0i} = 4U^i + \nu c\chi_{,0i} \, . \tag{6.19}$$

In this approximation one has also the density relation

$$\rho^* = \rho\,[1 - c^{-2}(\tfrac{1}{2}v^2 + 3U) + \ldots]. \tag{6.20}$$

Now it is possible to return to (6.12) to find the Lagrangian L of the post–Newtonian N–body problem in harmonic coordinates (or SPN coordinates considering that function a_0 does not affect the post–Newtonian equations of motion). By using expansions (6.3)–(6.5) in the post–Newtonian approximation and taking into account (6.13) one finds in result of some algebraic manipulations (Brumberg, 1972) the post–Newtonian Lagrangian

$$\begin{aligned} L = \int \rho \Big\{ & \big[\tfrac{1}{2}v^2 + U + c^{-2}(\tfrac{1}{8}v^4 + \tfrac{3}{2}v^2 U + \tfrac{1}{2}U^2 - 4U^i v^i - \tfrac{1}{2}\underset{4}{h}_{00})\big] + \\ & + \big[-\tfrac{1}{2}U + c^{-2}(-U^2 + 2v^i U^i - \tfrac{1}{2}cv^i\chi_{,0i} + \tfrac{1}{2}\underset{4}{h}_{00})\big] + \\ & + \Pi\big[-1 + c^{-2}(\tfrac{1}{2}v^2 + U)\big]\Big\} d^3x \, . \end{aligned} \tag{6.21}$$

As mentioned above the expression in the first square brackets resulting from ds/dx^0 represents the Lagrangian of the geodesic motion equations of a test particle in a given field. The expression in the second square brackets results from the first term of the integrand of (6.12). Adding of these two expressions results in remarkable simplification of the Lagrangian of the post–Newtonian N–body problem, i.e. cancelling $\underset{4}{h}_{00}$ terms. The terms dependent on the internal structure of the bodies may be treated as in (Brumberg, 1972) by using

$$\rho\Pi = \rho\Pi^* - c^{-2}p\,(\tfrac{1}{2}v^2 + 3U) + \ldots, \qquad \Pi^* = -\frac{p}{\rho} + \int_0^p \frac{dp}{\rho^*}\,, \tag{6.22}$$

Π^* being the Newtonian value of Π. Taking into account that such internal structure terms result only in re–definition of parameters (e.g., introducing 'effective' masses, etc.) and omitting for the sake of simplicity such terms we have finally

$$L = \sum_A \int_A \rho\,[\tfrac{1}{2}v^2 + \tfrac{1}{2}U + c^{-2}(\tfrac{1}{8}v^4 + \tfrac{3}{2}v^2 U - \tfrac{1}{2}U^2 - 2U^i v^i - \tfrac{1}{2}cv^i\chi_{,0i})]\,d^3x \tag{6.23}$$

with summation over all gravitating bodies labelled $A, B, C, \ldots$.

Integration in (6.23) may be performed by the standard technique by Fock separating U, U^i, χ into internal and external parts with respect to body A and expanding the external parts in the vicinity of body A (Fock, 1955; Brumberg, 1972). In particular, for the model of point masses there results the well–known Lagrangian of the EIH (Einstein–Infeld–Hoffman) equations

$$\begin{aligned} L = \sum_A \Bigg\{ & \tfrac{1}{2}m_A\mathbf{v}_A^2 + \tfrac{1}{2}G\sum_{B\neq A}\frac{m_A m_B}{r_{AB}} + c^{-2}\Bigg[\tfrac{1}{8}m_A(\mathbf{v}_A^2)^2 + \tfrac{1}{4}G\sum_{B\neq A}\frac{m_A m_B}{r_{AB}}\Big(3\mathbf{v}_A^2 + \\ & + 3\mathbf{v}_B^2 - 7\mathbf{v}_A\mathbf{v}_B - (\mathbf{v}_A\mathbf{r}_{AB})(\mathbf{v}_B\mathbf{r}_{AB})\frac{1}{r_{AB}^2}\Big) - \tfrac{1}{4}G^2\sum_{B\neq A}\frac{m_A m_B(m_A + m_B)}{r_{AB}^2} - \\ & - \tfrac{1}{6}G^2\sum_{B\neq A}\sum_{C\neq A,B} m_A m_B m_C\Big(\frac{1}{r_{AB}r_{AC}} + \frac{1}{r_{BA}r_{BC}} + \frac{1}{r_{CA}r_{CB}}\Big)\Bigg]\Bigg\} \end{aligned} \tag{6.24}$$

with $\mathbf{r}_{AB} = \mathbf{x}_A - \mathbf{x}_B$, $\mathbf{v}_A = \dot{\mathbf{x}}_A$, $\mathbf{x}_A$ denoting the position vector of body A. Assuming in the EIH equations one of the N masses to be zero one gets the equations of the restricted $N-1$ problem (motion of a test particle in the field of $N-1$ gravitating bodies).

In present practice the non–point structure of the bodies is taken into account only at the Newtonian level in the Newtonian parts of the post–Newtonian equations of motion. To take into account the non–point structure at the post–Newtonian level (mainly for advanced research study) it is to be reminded that such body characteristics (rotation velocity, multipole moments, etc.) should be considered in a body reference system. In so doing it is sufficient to use the post–Newtonian theory of reference system transformations in the equations obtained in a global reference system (e.g., Bretagnon and Brumberg, 2003) not demanding $\underset{4}{h}_{00}$ (both in a global system and a local body–related system). It involves relativistic contributions to the parameters occurring in U and U^i. The post–Newtonian equations of rotation of bodies also may be derived from the variational principle. Now it is possible to improve the first results obtained in this way in (Michalska, 1960a,b).

To conclude it is to be noted once again that for high accuracy research (strong gravitational field, high accurate relationship between proper time and coordinate time, etc.) one should know $\underset{4}{h}_{00}$ (and even more advanced terms) in the expansions of the metric coefficients. But the results obtained half a century ago (Infeld, 1957) and reminded here clearly show that for post–Newtonian celestial mechanics and astrometry there is no need in it.

It might seem that this conclusion is in contradiction to the IAU (2000) Resolution B1 (IAU, 2001) on reference systems and time scales claiming the necessity of $\underset{4}{h}_{00}$ term. As stated above this term is relevant for the geodesic principle and PNA techniques implicitly envisaged by this resolution. In this respect it is of interest that the original EIH technique (unsurpassed by its elegance) demands not only $\underset{4}{h}_{00}$ but also $\underset{4}{h}_{ij}$ and $\underset{5}{h}_{0i}$. No one resolution can restrict the search for more compact and more efficient techniques to solve problems.

It is of interest also that one of the principal PPN (Parametrized Post–Newtonian) formalism parameters β enters into $\underset{4}{h}_{00}$ (extending formally the Eddington–Robertson metric for one–body problem). If the basic equations of relativistic celestial mechanics were derived from the very beginning without this term then this parameter might not been introduced into equations of motion at all. The GRT variational principle underlines once again the compactness of GRT as compared with alternative theories of gravitation.

Needless to say, all practical GRT tools such as EIH planetary equations, equations of Earth's satellite and lunar motion, Earth's rotation equations, algorithms of reference systems transformations, etc., are well known now but it is of importance that all these tools might be obtained based on a very simple metric.

The foregoing discussion enables one to answer positively the question put in the title and to formulate

Conjecture 5: The linearized (weak–field) GRT metric is sufficient for post–Newtonian celestial mechanics and astrometry permitting to obtain the post–Newtonian equations of motion and rotation of celestial bodies.

Combination of Conjectures 4 and 5 results in rather simple structure of post–Newtonian celestial mechanics. Planetary motions in the solar system are investigated in BRS using the linearized GRT metric and EIH equations. The problems of satellite motion and planetary rotation may be treated using the Newtonian equations added by relativistic terms due to the primary (within the one–body problem) and performing the transformation between BRS and local (primary) RS to take into account the relativistic indirect third–body perturbations.

To conclude this section it may be noted that relativistic celestial mechanics being much younger than Newtonian celestial mechanics and anticipating soon its first centenary has in its history many unjustly forgotten ideas. One of them was considered is this section. Another interesting idea by Infeld related with the derivation of the EIH equations is the possibility to present the GRT N–body equations in purely Newtonian form by using adequately chosen coordinates (Infeld, 1953; Brumberg, 1972). Such coordinates are not quasi–Galilean coordinates (involving, in particular, the coordinate function $\underset{0}{a}$ of the first order) and the description of astronomical observations (light propagation) in such coordinates might be rather cumbersome. Nevertheless, the exploration of this idea is of interest. The currently employed relativistic hierarchy of quasi–Galilean reference systems is very effective practical tool but in no way this is the only possible way of research. The present developments in relativistic celestial mechanics and astrometry seems to be too pragmatic and self–restricting, especially if supported by IAU resolutions (Brumberg and Groten, 2001).

7. Solar system dynamics in the cosmological background

The motion of the solar system bodies is regarded in relativistic celestial mechanics (and so more in Newtonian celestial mechanics) almost always under assumption of the isolated existence of the solar system, i.e. by neglecting the cosmological background. This assumption is quite justified within the present observational results. But each time when there are some hints of discrepancies in the discussion of observations (e.g., time–quadratic terms in the planetary longitudes) the influence of the cosmological background is reminded as one of the possible sources of such discrepancies.

The investigation of this question was started by McVittie (1933) and continued by Järnefelt (1940, 1942) by considering the one–body problem in the expanding Universe (the Schwarzschild problem in the cosmological background). The problem was treated as a mathematical one. The corresponding results were formulated in terms of coordinates rather than in terms of physically measurable quantities (the techniques of relativistic reduction of observations were not elaborated by that time). In monograph (Brumberg, 1991) this problem was outlined in relation with the equations in variations for the spherically symmetrical background metric but the inadequate choice of coordinates involved the unnecessary mathematical difficulties. The satisfactory treatment of this problem based on the discussion of observations and the use of an adequate mathematical techniques seemed to be done for the first time in (Krasinsky and Brumberg, 2004). The conclusion is that within the present precision level the cosmological background (the expansion of the Universe) does not affect the motion of the planets and the reason of the possible observational discrepancies might be anything else. This problem needs to be further investigated. We give below the key formulas for this investigation.

Equations in variations for the spherically symmetrical metric:

The starting point is the GRT (general relativity theory) field equations (Brumberg, 1991)

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = -\kappa(\mathcal{T}^{\mu\nu} + T^{\mu\nu}) \,, \tag{7.1}$$

where $\mathcal{T}^{\mu\nu}$ is the background field mass tensor, $T^{\mu\nu}$ is the perturbation field mass tensor, $G^{\mu\nu}$ denotes the Einstein tensor, Λ is the cosmological constant, and $\kappa = 8\pi G/c^2$, G being the gravitational constant. The metric form is represented by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7.2}$$

with background metric tensor $g_{\mu\nu}$ and perturbations $h_{\mu\nu}$. The background isotropic metric for $\Lambda = 0$, $T^{\mu\nu} = 0$ for the spherically symmetrical field can be represented by

$$\eta_{00} = A \,, \qquad \eta_{0m} = 0 \,, \qquad \eta_{mn} = -B\delta_{mn} \,, \tag{7.3}$$

$$\eta^{00} = \frac{1}{A} \,, \qquad \eta^{0m} = 0 \,, \qquad \eta^{mn} = -\frac{\delta_{mn}}{B} \,, \tag{7.4}$$

A and B being functions of the radial coordinate distance r and time t to be determined from the background field equations. The field equations (7.1) can be rewritten with Ricci tensor $R_{\mu\nu}$ in form

$$R_{\mu\nu} = -\kappa(\mathcal{T}^*_{\mu\nu} + T^*_{\mu\nu}) + \Lambda g_{\mu\nu} \tag{7.5}$$

with

$$\mathcal{T}^*_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{T} \,, \qquad T^*_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \,, \tag{7.6}$$

$\mathcal{T}$ and T being invariants of the mass tensors $\mathcal{T}^{\mu\nu}$ and $T^{\mu\nu}$, respectively. Therefore, the equations in variations for the field equations (7.5) read (Brumberg, 1991)

$$\delta R_{\mu\nu} = -\kappa T^*_{\mu\nu} - \kappa \delta \mathcal{T}^*_{\mu\nu} + \Lambda g_{\mu\nu} \tag{7.7}$$

with

$$\delta \mathcal{T}^*_{\mu\nu} = \mathcal{T}^*_{\mu\nu}(g_{\alpha\beta}) - \mathcal{T}^*_{\mu\nu}(\eta_{\alpha\beta}). \tag{7.8}$$

Under the coordinate conditions

$$h_{00,0} + h_{ss,0} - 2h_{0s,s} = 0\,, \qquad h_{00,m} - h_{ss,m} + 2h_{ms,s} = 0 \tag{7.9}$$

the equations in variations for the background metric (7.3) are reduced to

$$h_{00,ss} - h_{00,00} = 2L_{00}\,, \tag{7.10}$$

$$h_{0m,ss} = 2L_{0m}\,, \tag{7.11}$$

$$h_{mn,ss} - \frac{B}{A}h_{mn,00} = 2L_{mn} + \Big(\frac{B}{A} - 1\Big)h_{00,mn} - \frac{B}{A}(h_{0m,0n} + h_{0n,0m}) \tag{7.12}$$

with contraction

$$h_{rr,ss} + h_{00,ss} = 2L_{ss} + 2\frac{B}{A}L_{00} \tag{7.13}$$

and the right–hand member functions

$$L_{\mu\nu} = B(\kappa T^*_{\mu\nu} + \kappa\delta\mathcal{T}^*_{\mu\nu} - \Lambda g_{\mu\nu} + Q_{\mu\nu})\,, \tag{7.14}$$

$Q_{\mu\nu}$ being non–linear contributions in Ricci tensor components given by (4.3.24)– (4.3.26) of (Brumberg, 1991). Equations in variations (7.10)–(7.12) are to be solved by iterations with respect to $h_{\mu\nu}$. At each step of iteration the right–hand members $2L_{\mu\nu}$ are known. Then, equation (7.10) is the wave equation with constant coefficients. The equation (7.11) is the Poisson equation. Equation (7.12) has the form of the wave equation with variable coefficient B/A (for example, for the background Schwarzschild metric this coefficient may be reduced to a function of r alone). But for $A = B$ this equation simplifies to be also the wave equation with constant coefficients. That's why it is reasonable to use the background cosmological solution (7.3) in the conformally Galilean coordinates ensuring the condition $A = B$ (in contrast to the Robertson–Walker metric in comoving coordinates with $A = 1$ and B being a function of r and t).

The background field equations:

The background solution is constructed here for simplicity with the simplest mass tensor

$$\mathcal{T}^{\mu\nu} = \rho\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}\,, \tag{7.15}$$

ρ being the density of the matter (dust matter without pressure). With the aid of (7.3) and (7.4) it leads to

$$\mathcal{T}^*_{00}(\eta_{\alpha\beta}) = \rho A\left(\frac{1}{2} + B\frac{dx^k}{ds}\frac{dx^k}{ds}\right), \qquad \mathcal{T}^*_{0m}(\eta_{\alpha\beta}) = -AB\rho\frac{dx^0}{ds}\frac{dx^m}{ds},$$

$$\mathcal{T}^*_{mn}(\eta_{\alpha\beta}) = \rho B\left(\frac{1}{2}\delta_{mn} + B\frac{dx^m}{ds}\frac{dx^n}{ds}\right). \tag{7.16}$$

These expressions together with the components of the background field Ricci tensor (3.1.14) of (Brumberg, 1991) enable one to get the background field equations as follows:

$$A_{,ss} - \frac{1}{2A}A_{,s}A_{,s} + \frac{1}{2B}A_{,s}B_{,s} - 3B_{,00} + \frac{3}{2A}A_{,0}B_{,0} + \frac{3}{2B}B_{,0}B_{,0} = \\ = 2AB\kappa\rho\left(\frac{1}{2} + B\frac{dx^k}{ds}\frac{dx^k}{ds}\right), \tag{7.17}$$

$$-2B_{,0i} + \frac{1}{A}A_{,i}B_{,0} + \frac{2}{B}B_{,i}B_{,0} = -2AB^2\kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds}\,, \tag{7.18}$$

$$- B_{,ik} - \delta_{ik} B_{,ss} + \frac{1}{2A}\big(A_{,i} B_{,k} + A_{,k} B_{,i} - \delta_{ik} A_{,s} B_{,s} + \delta_{ik} B_{,0} B_{,0}\big) +$$
$$+ \frac{1}{2B}\big(3B_{,i} B_{,k} + \delta_{ik} B_{,s} B_{,s}\big) - \frac{B}{A}\left(A_{,ik} - \delta_{ik} B_{,00} + \frac{1}{2A}\delta_{ik} A_{,0} B_{,0} - \frac{1}{2A} A_{,i} A_{,k}\right) =$$
$$= B^2 \kappa\rho \left(\delta_{ik} + 2B \frac{dx^i}{ds}\frac{dx^k}{ds}\right). \tag{7.19}$$

The cosmological background (the isotropic models) in comoving coordinates ($A = 1$):

In comoving coordinates one has

$$A = 1, \qquad \frac{dx^0}{ds} = 1, \qquad \frac{dx^m}{ds} = 0\,. \tag{7.20}$$

Under these conditions the field equations (7.17)–(7.19) admit the solution

$$B = \frac{a^2}{\left(1 + \frac{1}{4}kr^2\right)^2}, \qquad r^2 = x^s x^s, \tag{7.21}$$

k being a real constant. Function a dependent only on time is determined by the equations

$$\frac{\ddot a}{a} = -\tfrac{1}{6} c^2 \kappa\rho\,, \tag{7.22}$$

$$a\ddot a + 2\dot a^2 + 2kc^2 = \tfrac{1}{2} a^2 c^2 \kappa\rho\,. \tag{7.23}$$

From these equations it follows

$$k\frac{c^2}{a^2} = \tfrac{1}{3} c^2 \kappa\rho - H^2\,, \qquad H = \frac{\dot a}{a}\,, \tag{7.24}$$

H being the Hubble constant. One has also

$$2a\ddot a + \dot a^2 + kc^2 = 0\,, \tag{7.25}$$

$$\dot\rho + 3H\rho = 0\,. \tag{7.26}$$

This Robertson–Walker metric reads:

$$ds^2 = dx^0 dx^0 - \frac{a^2}{\left(1 + \frac{1}{4}kr^2\right)^2} dx^s dx^s\,. \tag{7.27}$$

With a new "time" argument η

$$dx^0 = a d\eta\,, \qquad \dot\eta = \frac{c}{a}\,, \qquad H = \frac{c}{a^2}\frac{da}{d\eta} \tag{7.28}$$

it describes explicitly three well–known cosmological models (with arbitrary linear constant q):

Closed model: $k = 1$

$$a = 2q(1 - \cos\eta), \qquad t = \frac{2q}{c}(\eta - \sin\eta), \qquad H = \frac{c}{2q}\frac{\sin\eta}{(1 - \cos\eta)^2}, \tag{7.29}$$

Flat model: $k = 0$

$$a = q\eta^2\,, \qquad t = \frac{q}{3c}\eta^3\,, \qquad H = \frac{2c}{q}\eta^{-3}, \tag{7.30}$$

Open model: $k = -1$

$$a = 2q(\cosh\eta - 1), \qquad t = \frac{2q}{c}(\sinh\eta - \eta), \qquad H = \frac{c}{2q}\frac{\sinh\eta}{(\cosh\eta - 1)^2}. \tag{7.31}$$

The cosmological background (the isotropic models) in conformally Galilean coordinates ($A = B$): In this case the field equations (7.17)–(7.19) result in

$$A_{,ss} - 3A_{,00} + \frac{3}{A}A_{,0}A_{,0} = 2A^2\kappa\rho\left(\frac{1}{2} + A\frac{dx^k}{ds}\frac{dx^k}{ds}\right) \tag{7.32}$$

$$-2A_{,0i} + \frac{3}{A}A_{,i}A_{,0} = -2A^3\kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds} \tag{7.33}$$

$$-2A_{,ik} - \delta_{ik}A_{,ss} + \frac{3}{A}A_{,i}A_{,k} + \delta_{ik}A_{,00} = A^2\kappa\rho\left(\delta_{ik} + 2A\frac{dx^i}{ds}\frac{dx^k}{ds}\right) \tag{7.34}$$

admitting the general expression for all three values of k

$$A = \left(1 - \frac{q}{d}\right)^4 + \left(\frac{x^0}{q}\right)^4\delta_{k0}\,, \qquad d = \sqrt{k(x^s x^s - x^0 x^0)} + q\delta_{k0}\,, \qquad q = \text{const}\,. \tag{7.35}$$

The expression of the Hubble constant for this case reads

$$H = \sqrt{-k}\,\frac{c}{d}\frac{1 + \frac{q}{d}}{\left(1 - \frac{q}{d}\right)^3} + \frac{2cq^2}{x^{0\,3}}\delta_{k0}\,. \tag{7.36}$$

Indeed, taking the derivatives of A from (7.35) and substituting them into the left–hand members of (7.32)–(7.34) one gets for $k \neq 0$

$$-12\left(1 - \frac{q}{d}\right)^2 k\frac{q}{d^3}\left(1 - 2k\frac{r^2}{d^2}\right) = A^2\kappa\rho\left(1 + 2A\frac{dx^k}{ds}\frac{dx^k}{ds}\right),$$

$$-24\left(1 - \frac{q}{d}\right)^2 k^2\frac{q}{d^5}x^0 x^i = -2A^3\kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds}\,,$$

$$-12\left(1 - \frac{q}{d}\right)^2 k\frac{q}{d^3}\delta_{ik} + 24\left(1 - \frac{q}{d}\right)^2 k^2\frac{q}{d^5}x^i x^k = A^2\kappa\rho\left(\delta_{ik} + 2A\frac{dx^i}{ds}\frac{dx^k}{ds}\right).$$

These equations are satisfied by the solution (7.35) for A together with

$$\kappa\rho = -k\frac{12q}{d^3\left(1 - \frac{q}{d}\right)^6}\,, \qquad \frac{dx^\mu}{ds} = \sqrt{-k}\frac{x^\mu}{d\left(1 - \frac{q}{d}\right)^2} \tag{7.37}$$

determining the density and the velocity of the dust matter (for $k = -1$ this solution coincides with the Fock (1955) solution).

Applying the same procedure for $k = 0$ one gets from (7.32)–(7.34)

$$12 = \frac{x^{0\,6}}{q^4}\kappa\rho\left(1 + 2\frac{x^{0\,4}}{q^4}\frac{dx^k}{ds}\frac{dx^k}{ds}\right),$$

$$0 = \kappa\rho\frac{dx^0}{ds}\frac{dx^i}{ds}\,,$$

$$12\delta_{ik}\frac{x^{0\,2}}{q^4} = \frac{x^{0\,8}}{q^8}\kappa\rho\left(\delta_{ik} + 2\frac{x^{0\,4}}{q^4}\frac{dx^i}{ds}\frac{dx^k}{ds}\right).$$

One comes again to the solution for A from (7.35) together with

$$\kappa\rho = 12\frac{q^4}{x^{0\,6}}\,. \tag{7.38}$$

This value of ρ corresponds to the critical density

$$\kappa\rho = \frac{3H^2}{c^2}\,, \tag{7.39}$$

as seen from (7.24).

In conformally Galilean coordinates the equation (7.26) takes the form

$$\dot{\rho} = -3H\rho\,\frac{ds}{dx^0}\,, \tag{7.40}$$

where

$$\left(\frac{ds}{dx^0}\right)^2 = \frac{A}{1 + A\frac{dx^k}{ds}\frac{dx^k}{ds}}\,. \tag{7.41}$$

In dealing with the conformally Galilean metric one may note a useful relation

$$\dot{d} = c\,\frac{ds}{dx^0}\,\frac{\sqrt{-k}}{\left(1-\frac{q}{d}\right)^2}\,. \tag{7.42}$$

Reduction of the Robertson–Walker metric to the conformally Galilean form:
Two different algorithms of such reduction are given below. The first one similar to the Fock (1955) representation for the open model involves complex coordinates for the closed model. The second algorithm involving only real coordinates is based on problem 19.8 of (Lightman et al,, 1975).

Algorithm 1:
Introducing spherical coordinates by means of

$$x^1 = r\cos\varphi\sin\theta\,,\quad x^2 = r\sin\varphi\sin\theta\,,\quad x^3 = r\cos\theta$$

one can represent the Robertson–Walker metric in form

$$ds^2 = a^2(\eta)\left[d\eta^2 - \frac{1}{\left(1+\frac{1}{4}kr^2\right)^2}(dr^2 + r^2 d\Omega^2)\right] \tag{7.43}$$

with

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2\,.$$

The transformation of the radial coordinate

$$\Sigma = \frac{r}{1+\frac{1}{4}kr^2}\,,\qquad d\Sigma = \frac{1-\frac{1}{4}kr^2}{\left(1+\frac{1}{4}kr^2\right)^2}dr \tag{7.44}$$

implying

$$1 - k\Sigma^2 = \frac{\left(1-\frac{1}{4}kr^2\right)^2}{\left(1+\frac{1}{4}kr^2\right)^2}\,,\qquad 1+\frac{1}{4}kr^2 = \frac{2}{1+\sqrt{1-k\Sigma^2}}$$

transforms (7.43) into

$$ds^2 = a^2(\eta)\left(d\eta^2 - \frac{d\Sigma^2}{1-k\Sigma^2} - \Sigma^2 d\Omega^2\right)\,. \tag{7.45}$$

A new variable χ determined by differential relation

$$\frac{d\Sigma^2}{1-k\Sigma^2} = d\chi^2 \tag{7.46}$$

or in explicit form

$$\Sigma = \begin{cases} \sin\chi\,, & k = +1 \quad \text{(closed model)} \\ \chi\,, & k = \ \ 0 \quad \text{(flat model)} \\ \sinh\chi\,, & k = -1 \quad \text{(open model)} \end{cases} \tag{7.47}$$

enables one to rewrite (7.45) as

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\chi^2 - \Sigma^2(\chi)d\Omega^2\right). \tag{7.48}$$

The final reduction to the conformally Galilean metric

$$ds^2 = A(c^2dt^2 - dr^2 - r^2d\Omega^2) \tag{7.49}$$

is provided by the transformation

$$ct = \sqrt{-k}\, d\cos(\sqrt{k}\chi) + \eta\, d\, \delta_{k0}\,, \qquad r = d\Sigma\,, \qquad d = q\exp(\sqrt{-k}\eta\,, \tag{7.50}$$

implying $c^2dt^2 - dr^2 = d^2(d\eta^2 - d\chi^2)$ and

$$A = \frac{a^2}{d^2}, \qquad a = -kd\left(1 - \frac{q}{d}\right)^2 + q\eta^2\delta_{k0}\,, \tag{7.51}$$

q being a real constant. Needless to say, $\cos \mathrm{i}\,\chi = \cosh\chi$. It is easy to see that the form (7.49) with (7.51) is identical to the previous found solution with (7.35).

Algorithm 2:
Starting from (7.48) it is possible instead of (7.50) to transform η and χ by means of

$$u = \frac{1}{2}(\eta + \chi), \qquad v = \frac{1}{2}(\eta - \chi), \tag{7.52}$$

reducing (7.48) to

$$ds^2 = a^2(4du\, dv - \Sigma^2 d\Omega^2). \tag{7.53}$$

Now u and v are changed to new variables α and β in a similar manner

$$\alpha = g(u), \qquad \beta = g(v), \qquad u = f(\alpha), \qquad v = f(\beta) \tag{7.54}$$

so that

$$du = f'(\alpha)d\alpha\,, \qquad f'(\alpha) = [g'(u)]^{-1}\,,$$

accent denoting the derivative with respect to the corresponding argument. Hence, the metric (7.53) transforms to

$$ds^2 = a^2 f'(\alpha)f'(\beta)\left[4d\alpha d\beta - \frac{\Sigma^2}{f'(\alpha)f'(\beta)}d\Omega^2\right]. \tag{7.55}$$

The function g is to be chosen to provide the condition for the conformally Galilean metric

$$\frac{\Sigma^2}{f'(\alpha)f'(\beta)} = (\alpha - \beta)^2 \tag{7.56}$$

or more specifically

$$g'(u)g'(v)\Sigma^2 = [g(u) - g(v)]^2\,, \tag{7.57}$$

Σ being considered as dependent only on the difference $u - v$. For any $k = +1,\, 0,\, -1$ this equation is satisfied by

$$g(u) = \tan(\sqrt{k}\, u) + u\, \delta_{k0} \tag{7.58}$$

with evident replacing $\tan \mathrm{i}\, u = \tanh u$. The metric (7.55) becomes

$$ds^2 = \frac{a^2}{(1 + k\alpha^2)(1 + k\beta^2)}[4d\alpha d\beta - (\alpha - \beta)^2 d\Omega^2]. \tag{7.59}$$

Finally, the transformation

$$\alpha = \frac{1}{2q}(c\tilde{t} + \tilde{r}), \qquad \beta = \frac{1}{2q}(c\tilde{t} - \tilde{r}) \tag{7.60}$$

reduces (7.59) to the conformally Galilean metric of the type (7.49)

$$ds^2 = \tilde{A}(c^2 d\tilde{t}^2 - d\tilde{r}^2 - \tilde{r}^2 d\Omega^2) \tag{7.61}$$

with

$$\tilde{A} = \frac{a^2}{q^2 D}, \tag{7.62}$$

$$D = (1 + k\alpha^2)(1 + k\beta^2) = 1 + \frac{k}{2q^2}(c^2\tilde{t}^2 + \tilde{r}^2) + \frac{k^2}{16q^4}(c^2\tilde{t}^2 - \tilde{r}^2)^2 . \tag{7.63}$$

By using

$$\sigma = g(\eta) \tag{7.64}$$

there results

$$a = 2q\frac{\sigma^2}{\sqrt{1 + k\sigma^2}(1 + \sqrt{1 + k\sigma^2})}, \qquad \sigma = \frac{\dfrac{c\tilde{t}}{q}}{1 - k\dfrac{c^2\tilde{t}^2 - \tilde{r}^2}{4q^2}} . \tag{7.65}$$

The equivalence of (7.49) and (7.61) may be easily seen from the expressions of $\tilde{t}$, $\tilde{r}$ from the one part, and t, r from the other part, in terms of η and χ. By comparing these expressions one gets

$$\tilde{r} = \sqrt{D}\,\frac{q}{d}\,r \tag{7.66}$$

and

$$c^2 d\tilde{t}^2 - d\tilde{r}^2 = q^2 D(d\eta^2 - d\chi^2) = \frac{q^2 D}{d^2}(c^2 dt^2 - dr^2) \tag{7.67}$$

(this formula corrects a misprint in (12.51) of Brumberg, 2005).
Hence,

$$A = \frac{q^2 D}{d^2}\,\tilde{A} \tag{7.68}$$

demonstrating the equivalence of (7.59) and (7.61). In what follows the form (7.61) will be used with no specification for $\tilde{A}$ and with omitting tilde over $\tilde{A}$, $\tilde{t}$ and $\tilde{r}$.

Solar gravitational field at the cosmological background

Let the variations of the background gravitational field of the expanding Universe be caused by a spherical massive body (the Sun) located at the spatial origin $\mathbf{r} = (\mathbf{x^k}) = \mathbf{o}$. Investigating a quasi–circular motion of a test particle in this field (Schwarzschild problem at the cosmological background) one may see if the expansion of the Universe affects the motion of the Solar system bodies (Krasinsky and Brumberg, 2004). For this purpose it is sufficient to have the simplest, just quasi–Newtonian solution of equations (7.10)–(7.12) by restricting in (7.14) only by the first term, i.e. $L_{\mu\nu} = \kappa B T^*_{\mu\nu}$. Then the disturbing mass tensor $T^{\mu\nu}$ may be taken in the form

$$T^{\mu\nu} = \frac{\tilde{\rho}}{\sqrt{-g}}\frac{dx^0}{ds}\frac{dx^\mu}{dx^0}\frac{dx^\nu}{dx^0} \tag{7.69}$$

with the density

$$\tilde{\rho} = M\delta(\mathbf{r}), \tag{7.70}$$

$\delta(\mathbf{r})$ being delta–function (Infeld and Plebansky 1960). For the background metric (7.3) one easily finds

$$BT^*_{00} = \frac{1}{2}\sqrt{A}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\left(\frac{A}{B} + \frac{v^2}{c^2}\right),$$

$$BT^*_{0i} = -\sqrt{A}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2} \frac{v^i}{c}\,,$$

$$BT^*_{ij} = \frac{B}{\sqrt{A}}\tilde{\rho}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)^{-1/2}\left[\frac{1}{2}\left(\frac{A}{B} - \frac{v^2}{c^2}\right)\delta_{ij} + \frac{v^i}{c}\frac{v^j}{c}\right]. \tag{7.71}$$

In integrating the equations (7.10)–(7.12) for the fixed material point in the conformally Galilean background with $A = B$ one may put

$$L_{00} = 4\pi\sqrt{A}m\delta(\mathbf{r})\,, \qquad L_{0i} = 0\,, \qquad L_{ij} = 4\pi\sqrt{A}m\delta(\mathbf{r})\delta_{ij} \tag{7.72}$$

with

$$m = \frac{GM}{c^2}\,. \tag{7.73}$$

Hence, by neglecting the retardation terms one may present the approximate solution of (7.10)–(7.12) in form

$$h_{00} = -\frac{2m}{r}\sqrt{A}\,, \qquad h_{0i} = 0\,, \qquad h_{ij} = -\frac{2m}{r}\sqrt{A}\,\delta_{ij}\,, \tag{7.74}$$

resulting to

$$ds^2 = \left(A - \frac{2m}{r}\sqrt{A}\right)c^2dt^2 - \left(A + \frac{2m}{r}\sqrt{A}\right)dx^s dx^s\,. \tag{7.75}$$

Equations of motion of a test particle in the field (7.75) with the coordinate time t as an argument follow from equations (4.3.38) of (Brumberg, 1991) for the general metric (7.3). By retaining only the main terms one has

$$\ddot{x}^i = \frac{1}{2A}(\dot{x}^s\dot{x}^s - c^2)(A_{,i} + c^{-1}A_{,0}\dot{x}^i) - \frac{c^2}{2A}h_{00,i} - \frac{c^2}{2A^2}A_{,s}h_{ms} + \ldots \tag{7.76}$$

or else

$$\ddot{x}^i = -\frac{GM}{\sqrt{A}r^3}x^i - \frac{\dot{A}}{2A}\dot{x}^i\,. \tag{7.77}$$

In reducing (7.76) to (7.77) the last term in the right–hand member of (7.76) is neglected due to the presence of the derivatives $A_{,s}$ much smaller as compared with A. For the same reason $cA_{,0}$ in (7.77) is replaced just by $\dot{A}$. Considering the motion of the test particle for some limited interval of time $t - t_0$ one may use approximation

$$A = A_0 + \dot{A}_0(t - t_0)\,, \tag{7.78}$$

using as a small parameter the ratio $(t - t_0)/T$ where

$$T = \frac{4A_0}{\dot{A}_0}\,. \tag{7.79}$$

Then, equation (7.77) reads

$$\ddot{x}^i = -\frac{GM^*}{r^3}\left(1 - 2\frac{t - t_0}{T}\right)x^i - \frac{2}{T}\dot{x}^i\,, \qquad M^* = \frac{M}{\sqrt{A_0}}\,. \tag{7.80}$$

It is seen that the main effect of these equations may be interpreted as the variability of the constant of gravitation

$$\frac{\dot{G}}{G} = -\frac{2}{T}\,. \tag{7.81}$$

On the other hand, in polar coordinates in the plane of motion $x^3 = 0$

$$x^1 = r\cos\lambda\,, \qquad x^2 = r\sin\lambda$$

these equations read

$$\ddot{r} - r\dot{\lambda}^2 = -GM^*\left(1 - 2\frac{t-t_0}{T}\right)\frac{1}{r^2} - \frac{2}{T}\dot{r}\,, \qquad \frac{1}{r}\frac{d}{dt}(r^2\dot{\lambda}) = -\frac{2}{T}r\,\dot{\lambda}\,, \tag{7.82}$$

or else

$$r^2\dot{\lambda} = na^2\exp\left(-2\frac{t-t_0}{T}\right), \qquad \dot{\lambda} = n\left(\frac{a}{r}\right)^2\left(1-2\frac{t-t_0}{T}\right),$$

$$\ddot{r} = \frac{n^2a^4}{r^3}\left(1 - 4\frac{t-t_0}{T}\right) - \frac{n^2a^3}{r^2}\left(1-2\frac{t-t_0}{T}\right) - \frac{2}{T}\dot{r}\,, \tag{7.83}$$

a being an arbitrary constant with $n^2a^3 = GM^*$. These equations admit an approximate solution

$$r = a\left(1 - 2\frac{t-t_0}{T}\right), \qquad \dot{\lambda} = n\left(1 + 2\frac{t-t_0}{T}\right), \tag{7.84}$$

involving the quadratic term in the mean longitude

$$\delta\lambda = \frac{n}{T}(t-t_0)^2\,. \tag{7.85}$$

This is just a coordinate–form solution. To get the physically meaningful relativistic effects it is necessary in general to perform the relativistic reduction of observations using the solution of the equations of the light propagation. These equations follow again from equations (4.3.38) of (Brumberg, 1991) under the substitution

$$B\dot{x}^s\dot{x}^s = c^2A + c^2h_{00} + 2c\,h_{0s}\dot{x}^s + h_{rs}\dot{x}^r\dot{x}^s \tag{7.86}$$

resulted from the condition $ds^2 = 0$ for the light propagation. By restricting only by the main terms one has for $A = B$

$$\ddot{x}^m = \frac{GM}{\sqrt{A}r^3}\left(-2x^m + \frac{4}{c^2}\dot{x}^m\dot{x}^s x^s\right). \tag{7.87}$$

In our case it is sufficient it is sufficient to express the longitude λ of the moving particle in terms of the proper time τ of this particle. From (7.75) it is seen that within the first order with respect to the small parameter of this problem the proper time τ reads

$$\frac{d\tau}{dt} = \sqrt{A_0}\left(1 + 2\frac{t-t_0}{T}\right) \tag{7.88}$$

demonstrating that the quadratic term (the only one significant term) in the mean longitude (7.85) does not exist as a measurable effect in $d\lambda/d\tau$. It follows from this that within the present–day observational precision there is no observable effect in the motion of the solar system bodies due to the cosmological background (Krasinsky and Brumberg, 2004).

However, this treatment is only one of the initial steps toward the global problem of the influence of the cosmological background. In particular, it may be of interest to correlate this approach with the exact solution for the one–body problem in an expanding universe by McVittie (1933) and Järnefelt (1940, 1942) or to apply the equations in variations for the conformally Galilean background field ($A = B$) for more wide class of perturbations. The main question seems to be in checking

Conjecture 6: The cosmological background does not affect qualitatively the motion of the solar system bodies.

8. Conclusion

In looking through the issues of *Celestial Mechanics and Dynamical Astronomy* journal of the recent years it is seen that present celestial mechanics has not so much common with celestial mechanics of the second half of the last century. The aim of the present paper is to show that the problems and the techniques of that period are not quite exhausted and they still might be of interest and benefit even in new surroundings. It concerns both Newtonian and relativistic celestial mechanics.

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