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SOME TECHNIQUES FOR DETERMINING RELATIVISTIC PLANETARY PERTURBATIONS IN THE THEORIES OF MOTION OF THE MAJOR PLANETS

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Some Techniques for Determining Relativistic Planetary Perturbations in the Theories of Motion of the Major Planets

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Abstract Presently, the relativistic planetary perturbations in the motion of the major planets are taken into account only implicitly in numerical integrating the equations of motion. This paper suggests two different techniques to determine the relativistic planetary perturbations of the first order with respect to the planetary masses as explicit functions of time. The first technique implies the iterative solution of the planetary equations in form of the exponential series in the mean longitudes of the planets with polynomial in time coefficients. In such a way the relativistic planetary perturbations may be expressed in the form compatible with the VSOP87 solution of the Bureau des Longitudes. The key point here is to correct properly the value of the mean longitude at the zero epoch at each step of iterations. The second technique generalizes the general planetary theory for the relativistic planetary problem. The solution is presented here by the exponential series in the mean longitudes, the coefficients being the series in powers of slowly varying elements (separation of the fast and slow variables). The behaviour of the latter elements is governed by the autonomous secular system. This technique enables one to compute the relativistic planetary perturbations by analytical algorithms. More specifically, the paper presents the formulas to compute the relativistic planetary perturbations independent of planetary eccentricities and inclinations (the intermediary) as well as the matrices of the secular system responsible for the long–term evolution of the planetary orbits within the general relativity framework.

1. **Introduction**

The present paper is aimed to give algorithms to determine relativistic planetary perturbations in the motion of the major planets. Until recently it was sufficient to take into account in the planetary motions only the relativistic Schwarzschild perturbations caused by the action of the Sun. But the permanently increasing accuracy of the planetary theories demands to investigate the relativistic planetary perturbations. Presently, such perturbations are taken into account implicitly in numerical planetary ephemerides resulted from the numerical integration of the post–Newtonian equations of the planetary motion. However, the presentation of these perturbations as explicit functions of time may be of obvious interest. Semi–analytical theories of the motion of the major planets VSOP87 (Bretagnon and Francou, 1988) are based on the Newtonian planetary equations complemented by the relativistic Schwarzschild terms alone. Therefore, the relativistic perturbations are presented in these theories by the Schwarzschild terms proportional to the gravitational radius of the Sun $m_0 = GM_0/c^2$ (with the solar mass M_0, gravitational constant G and light velocity c) and the indirect planetary terms proportional to $m_0 M_i$ (M_i — being the planetary masses) and caused by the interaction of the Newtonian planetary perturbations and Schwarzschild terms. Our aim is to take into account the relativistic direct planetary perturbations in the equations of motion and to determine eventually all relativistic terms proportional to $\mu\sigma$, the small parameters $\mu = 10^{-3}$ and $\sigma = 10^{-8}$ characterize, respectively, the smallness of the planetary masses with respect to the mass of the Sun and the smallness of the relativistic effects in the solar system (the ratio of the solar gravitational radius to the astronomical unit of length). The algorithms developed below enable one to find these terms both in the form of the classic planetary theories like VSOP87 (trigonometric series in multiples of the planetary mean longitudes with time–polynomial coefficients) and in the form of the general planetary theory (Brumberg, 1995) (trigonometric series in multiples of the planetary mean longitudes with polynomial coefficients in terms of the slowly changing variables whose behaviour is described by an autonomous secular system).

2. Relativistic equations of motion of the first order

To start with, we take the heliocentric equations of motion of N major planets (Brumberg, 1991)

$$\begin{aligned}\ddot{\mathbf{R}}_i =& - G(M_0 + M_i)\frac{\mathbf{R}_i}{R_i^3} + \sum_{j\neq i} GM_j\left(\frac{\mathbf{R}_j - \mathbf{R}_i}{R_{ij}^3} - \frac{\mathbf{R}_j}{R_j^3}\right) + \\ &+ (m_0 A_{i0} + m_i A_{0i})\mathbf{R}_i + (m_0 B_{i0} + m_i B_{0i})\dot{\mathbf{R}}_i + \\ &+ \sum_{j\neq i} m_j \left[A_{ij}\,(\mathbf{R}_i - \mathbf{R}_j) + A_{0j}\mathbf{R}_j + B_{ij}(\dot{\mathbf{R}}_i - \dot{\mathbf{R}}_j) + B_{0j}\dot{\mathbf{R}}_j\right] , \\ &i,\, j = 1,\, \dots\, ,\, N\, ,\end{aligned} \tag{2.1}$$

with

$$\mathbf{R}_i = \mathbf{x}_i - \mathbf{x}_0\, , \qquad \mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j\, , \qquad m_i = \frac{GM_i}{c^2}\, . \tag{2.2}$$

The formally heliocentric equations (2.1) are just the differences of the barycentric equations of motion of planet i and the Sun described in the relativistic harmonic reference system $(ct,\, \mathbf{x})$ with the barycentric coordinate time $t = TCB$ as their argument. The expressions of the coefficients of the right–hand members of these equations are given in (Brumberg, 1991). The barycentric coordinates of the Sun $\mathbf{x}_0$ may be determined from the integral

$$M\mathbf{R} = -\frac{1}{2c^2}\left[M_0\left(\dot{\mathbf{x}}_0^2 - \sum_{j=1}^N \frac{GM_j}{r_{0j}}\right)\mathbf{x}_0 + \sum_{i=1}^N M_i\left(\dot{\mathbf{x}}_i^2 - \frac{GM_0}{r_{0i}} - \sum_{j\neq i}\frac{GM_j}{r_{ij}}\right)\mathbf{x}_i\right], \tag{2.3}$$

$\mathbf{R}$ being the position vector of the Newtonian center of mass of the Sun and all planets

$$\mathbf{R} = \frac{1}{M}\left(M_0\mathbf{x}_0 + \sum_{i=1}^N M_i\mathbf{x}_i\right) \tag{2.4}$$

and

$$\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j\,, \qquad M = M_0 + \sum_{i=1}^{N} M_i\,. \tag{2.5}$$

Within the practically admissible accuracy of the first order with respect to the planetary masses there results

$$M\mathbf{R} = -\frac{1}{2c^2}\sum_{i=1}^{N} M_i \left(\dot{\mathbf{R}}_i^2 - \frac{GM_0}{R_i}\right)\mathbf{R}_i\,, \tag{2.6}$$

$$\mathbf{x}_0 = -\frac{1}{M}\sum_{i=1}^{N} M_i \left[1 + \frac{1}{2c^2}\left(\dot{\mathbf{R}}_i^2 - \frac{GM_0}{R_i}\right)\right]\mathbf{R}_i\,. \tag{2.7}$$

The coefficients A_{i0} and B_{i0} are demanded within the first order accuracy whereas for the coefficients A_{ij}, B_{ij}, A_{0i} and B_{0i} the zero order accuracy is sufficient. From the rigorous expressions of these coefficients given in (Brumberg, 1991) one has within the demanded accuracy

$$\begin{aligned} A_{ij} = &\frac{1}{R_{ij}^3}\left[4\dot{\mathbf{R}}_i\dot{\mathbf{R}}_j - \dot{\mathbf{R}}_i^2 - 2\dot{\mathbf{R}}_j^2 + \frac{3}{2R_{ij}^2}(\mathbf{R}_{ij}\dot{\mathbf{R}}_j)^2 + \frac{4GM_0}{R_i} + \frac{GM_0}{R_j} - \right.\\ &\left. - \frac{GM_0}{2R_j^3}(\mathbf{R}_{ij}\mathbf{R}_i)\right] + \frac{GM_0}{R_j^3}\left(\frac{4}{R_{ij}} - \frac{7}{2R_i}\right), \end{aligned} \tag{2.8}$$

$$B_{ij} = \frac{1}{R_{ij}^3}\left[4(\mathbf{R}_{ij}\dot{\mathbf{R}}_{ij}) + (\mathbf{R}_{ij}\dot{\mathbf{R}}_j)\right], \tag{2.9}$$

$$A_{0i} = \frac{1}{R_i^3}\left[-2\dot{\mathbf{R}}_i^2 + \frac{3}{2R_i^2}(\mathbf{R}_i\dot{\mathbf{R}}_i)^2 + \frac{5GM_0}{R_i}\right], \tag{2.10}$$

$$B_{0i} = \frac{3}{R_i^3}(\mathbf{R}_i\dot{\mathbf{R}}_i), \tag{2.11}$$

$$\begin{aligned} A_{i0} = &\frac{1}{R_i^3}\left(-\dot{\mathbf{R}}_i^2 + 4\frac{GM_0}{R_i} + 2(\dot{\mathbf{R}}_i\dot{\mathbf{x}}_0) + 5\frac{GM_i}{R_i}\right) + \\ &+ \sum_{j\neq i} GM_j\left(\frac{1}{R_i^3R_j} + \frac{7}{2}\frac{1}{R_iR_j^3} + \frac{1}{2}\frac{\mathbf{R}_i\mathbf{R}_j}{R_i^3R_j^3} + \frac{4}{R_i^3R_{ij}} - \frac{7}{2}\frac{1}{R_j^3R_{ij}}\right), \end{aligned} \tag{2.12}$$

$$B_{i0} = \frac{1}{R_i^3}\left[4(\mathbf{R}_i\dot{\mathbf{R}}_i) + (\mathbf{R}_i\dot{\mathbf{x}}_0)\right]\,. \tag{2.13}$$

The right–hand members of equations (2.1) contain two small dimensionless parameters

$$\mu = 10^{-3}\,, \qquad \sigma = 10^{-8}. \tag{2.14}$$

The first parameter is to characterize the smallness of the planetary masses with respect to the mass of the Sun. Together with the parameter μ one makes use of $N \times N$ matrix $\kappa = ||\kappa(i,j)||$ with the elements

$$\mu\kappa_{ii} = \frac{M_i}{M_0}\,, \qquad \mu\kappa_{ij} = \frac{M_j}{M_0 + M_i}\,, \quad i \neq j\,. \tag{2.15}$$

Parameter σ is to characterize the smallness of the relativistic effects in the solar system. Introducing a characteristic length A (nearly equivalent to the astronomical unit of length) one has

$$\sigma = \frac{GM_0}{c^2 A}\,. \tag{2.16}$$

The employed below the semi–major axes of the planetary orbits a_i are determined from the observed values of the mean motions n_i by means of the third Kepler law

$$n_i^2 a_i^3 = G(M_0 + M_i)\,. \tag{2.17}$$

The coefficients (2.13) and (2.14) may be now rewritten as follows:

$$A_{i0} = A'_{i0} + \mu A^*_{i0}\,, \qquad B_{i0} = B'_{i0} + \mu B^*_{i0} \tag{2.18}$$

with

$$A'_{i0} = \frac{1}{R_i^3}\left(-\dot{\mathbf{R}}_i^2 + 4\frac{n_i^2 a_i^3}{R_i}\right), \tag{2.19}$$

$$B'_{i0} = \frac{4}{R_i^3}(\mathbf{R}_i\dot{\mathbf{R}}_i)\,, \tag{2.20}$$

$$\begin{aligned} A^*_{i0} =& \frac{\kappa_{ii}}{R_i^3}\left(-2\dot{\mathbf{R}}_i^2 + \frac{GM_0}{R_i}\right) - 2\frac{\dot{\mathbf{R}}_i}{R_i^3}\sum_{j\neq i}\kappa_{jj}\dot{\mathbf{R}}_j + \\ &+ n_i^2 a_i^3 \sum_{j\neq i}\kappa_{ij}\left(\frac{1}{R_i^3 R_j} + \frac{7}{2}\frac{1}{R_i R_j^3} + \frac{1}{2}\frac{\mathbf{R}_i\mathbf{R}_j}{R_i^3 R_j^3} + \frac{4}{R_i^3 R_{ij}} - \frac{7}{2}\frac{1}{R_j^3 R_{ij}}\right), \end{aligned} \tag{2.21}$$

$$B^*_{i0} = -\kappa_{ii}\frac{\mathbf{R}_i\dot{\mathbf{R}}_i}{R_i^3} - \frac{\mathbf{R}_i}{R_i^3}\sum_{j\neq i}\kappa_{jj}\dot{\mathbf{R}}_j\,. \tag{2.22}$$

Needless to say, in using the Newtonian value for $\dot{\mathbf{x}}_0$ the quantity M is replaced herewith by M_0 within the first order accuracy. In virtue of (2.17) the coefficient (2.19) depends implicitly on M_i. The splitting (2.18) into two parts is somewhat arbitrary. The specific expressions (2.19) and (2.21) are chosen for the sake of convenience. The initial equations (2.1) take now the form

$$\ddot{\mathbf{R}}_i = \mathbf{F}_i\,, \qquad \mathbf{F}_i = \mathbf{F}_K^{(i)} + \mathbf{F}_N^{(i)} + \mathbf{F}_S^{(i)} + \mathbf{F}_R^{(i)} \tag{2.23}$$

with Keplerian terms

$$\mathbf{F}_K^{(i)} = -n_i^2 a_i^3 \frac{\mathbf{R}_i}{R_i^3}\,, \tag{2.24}$$

Newtonian perturbing terms

$$\mathbf{F}_N^{(i)} = n_i^2 a_i^3 \mu \sum_{j\neq i} \kappa_{ij} \left(\frac{\mathbf{R}_j - \mathbf{R}_i}{R_{ij}^3} - \frac{\mathbf{R}_j}{R_j^3} \right), \tag{2.25}$$

relativistic Schwarzschild terms

$$\mathbf{F}_S^{(i)} = \sigma A (A'_{i0}\mathbf{R}_i + B'_{i0}\dot{\mathbf{R}}_i) \tag{2.26}$$

and relativistic planetary perturbing terms

$$\begin{aligned}\mathbf{F}_R^{(i)} =& \mu\sigma A \Bigg\{ (A^*_{i0} + \kappa_{ii} A_{0i})\mathbf{R}_i + (B^*_{i0} + \kappa_{ii} B_{0i})\dot{\mathbf{R}}_i + \\ &+ \sum_{j\neq i} \kappa_{jj} \Big[A_{ij}(\mathbf{R}_i - \mathbf{R}_j) + A_{0j}\mathbf{R}_j + B_{ij}(\dot{\mathbf{R}}_i - \dot{\mathbf{R}}_j) + B_{0j}\dot{\mathbf{R}}_j \Big] \Bigg\}. \end{aligned} \tag{2.27}$$

On a matter of concern, the first order relativistic planetary perturbations proportional to $\mu\sigma$ are due to the direct perturbing terms $\mathbf{F}_R^{(i)}$ in the right–hand members of the equations of motion and to the interaction of perturbations caused by the terms $\mathbf{F}_N^{(i)}$ and $\mathbf{F}_S^{(i)}$. Clearly, to determine these perturbations one may use a variety of the theory of perturbations techniques. If the solution of the equations (2.23) for $\mathbf{F}_R^{(i)} = 0$ is known, e.g. in form of the VSOP solution, then one can add to this solution the perturbations due to $\mathbf{F}_R^{(i)}$ by means of the iteration procedure of the next section. On the other hand, it seems reasonable to investigate these perturbations in a more sophisticated manner extending the general planetary theory (Brumberg, 1995) for relativistic equations (2.23). This technique is exposed starting with the section 4.

3. Iterative solution of the perturbed two–body problem

The initial equations of the perturbed two–body problem are of the form

$$\ddot{x} + \frac{GMx}{r^3} = X \tag{3.1}$$

in combination with two similar equations for the coordinates y and z (with the right–hand members Y and Z, respectively). Performing the transformation

$$x + \mathrm{i}\, y = a(1-p)\exp \mathrm{i}\,\lambda\,, \qquad z = aw\,, \tag{3.2}$$

with

$$\lambda = nt + \varepsilon\,, \qquad n^2 a^3 = GM\,, \tag{3.3}$$

one gets the new equations

$$\ddot{p} + 2\,\mathrm{i}\,n\dot{p} - \frac{3}{2}n^2(p+q) = n^2 P\,, \tag{3.4}$$

$$\ddot{w} + n^2 w = n^2 W \tag{3.5}$$

with the right–hand members

$$P = -\frac{3}{2}(p+q) + \left(\frac{a^3}{r^3} - 1\right)(1-p) - \frac{1}{n^2 a}(X + \mathrm{i}\,Y)\exp(-\,\mathrm{i}\,\lambda), \tag{3.6}$$

$$W = -\left(\frac{a^3}{r^3} - 1\right)w + \frac{1}{n^2 a}Z\,, \tag{3.7}$$

with $q = \bar{p}$. Here and below the bar means a complex conjugate quantity. These equations can be solved by iterations as described in section 7.2 of (Brumberg, 1995). Indeed, if some approximate solution of the equations (3.4) and (3.5) is substituted into the right–hand members of these equations then a more accurate solution can be found by quadratures resulted from the iterative equations

$$\begin{aligned} p =&\,\mathrm{i}\,n\left[\frac{3}{4}\exp(-\,\mathrm{i}\,\lambda)\int(3P^+ - \bar{P}^+)\exp\mathrm{i}\,\lambda\,dt + \right.\\ &\left. + \frac{1}{4}\exp\mathrm{i}\,\lambda\int(3\bar{P}^+ - P^+)\exp(-\,\mathrm{i}\,\lambda)\,dt - 2\int P^+\,dt\right] - \\ &- \frac{3}{2}n_i^2\int\!\!\int(P^+ - \bar{P}^+)\,dt\,dt - \frac{1}{3}P^* + \tilde{p}\,, \end{aligned} \tag{3.8}$$

$$w = \frac{1}{2}\,\mathrm{i}\,n\left[\exp(-\,\mathrm{i}\,\lambda)\int W\exp\mathrm{i}\,\lambda\,dt - \exp\mathrm{i}\,\lambda\int W\exp(-\,\mathrm{i}\,\lambda)\,dt\right] + \tilde{w}\,. \tag{3.9}$$

Here P^* stands for the constant (not dependent on time) part of P, so that

$$P = P^* + P^+\,. \tag{3.10}$$

The derivatives $\dot{p}$ and $\dot{w}$ are determined by

$$\begin{aligned} \dot{p} =&\,n^2\left[\frac{3}{4}\exp(-\,\mathrm{i}\,\lambda)\int(3P^+ - \bar{P}^+)\exp\mathrm{i}\,\lambda\,dt - \right.\\ &\left. - \frac{1}{4}\exp\mathrm{i}\,\lambda\int(3\bar{P}^+ - P^+)\exp(-\,\mathrm{i}\,\lambda)\,dt - \frac{3}{2}\int(P^+ - \bar{P}^+)\,dt\right] + \dot{\tilde{p}}\,, \end{aligned} \tag{3.11}$$

$$\dot{w} = \frac{1}{2}n^2\left[\exp(-\,\mathrm{i}\,\lambda)\int W\exp\mathrm{i}\,\lambda\,dt + \exp\mathrm{i}\,\lambda\int W\exp(-\,\mathrm{i}\,\lambda)\,dt\right] + \dot{\tilde{w}}\,. \tag{3.12}$$

The functions $\tilde{p}$, $\tilde{w}$, $\dot{\tilde{p}}$ and $\dot{\tilde{w}}$ represent the general solution of the homogeneous equations related to (3.4) and (3.5). This general solution is described by the expressions

$$p = A \exp \mathrm{i}\,\lambda - 3\bar{A} \exp(-\,\mathrm{i}\,\lambda) + \mathrm{i}\; n(B + 3Ct) - 2C\,, \tag{3.13}$$

$$w = D \exp \mathrm{i}\,\lambda + \bar{D} \exp(-\,\mathrm{i}\,\lambda)\,, \tag{3.14}$$

$$\dot{p} = \mathrm{i}\; n\big[A \exp \mathrm{i}\,\lambda + 3\bar{A} \exp(-\,\mathrm{i}\,\lambda) + 3C\big], \tag{3.15}$$

$$\dot{w} = \mathrm{i}\; n\big[D \exp \mathrm{i}\,\lambda - \bar{D} \exp(-\,\mathrm{i}\,\lambda)\big]\,. \tag{3.16}$$

Here A and D are complex arbitrary constants, B and C are real constants. In the expressions of $\tilde{p}$, $\tilde{w}$, $\dot{\tilde{p}}$ and $\dot{\tilde{w}}$ the constants B and C are annulled. The constants A and D together with the constants n and ε present a full set of six real arbitrary constants in the solution of the equations (3.4) and (3.5). The meaning of the constants A and D is evident from the initial terms of their expressions in terms of the traditional Kepler elements e, i, π, Ω

$$A = -\frac{1}{2} e \exp(-\,\mathrm{i}\,\pi)\,, \qquad D = -\frac{1}{2}\; \mathrm{i}\, \sin i \exp(-\,\mathrm{i}\,\Omega)\,. \tag{3.17}$$

Performing iterations by means of (3.8) and (3.9) is possible provided that the constant part P^* of P is real, i.e.

$$\Im(P^*) = 0\,. \tag{3.18}$$

Let's consider new equations

$$\ddot{x}' + \frac{GMx'}{r'^3} = X' \tag{3.19}$$

with the right–hand members

$$X' = X + \delta X\,. \tag{3.20}$$

In case of the VSOP solution the right–hand members X involve the Newtonian planetary perturbations and Schwarzschild terms whereas the additive right–hand members δX represent the relativistic planetary disturbing terms. Similar to (3.2) and (3.3) the transformation of the variables

$$x' + \mathrm{i}\, y' = a(1 - p') \exp \mathrm{i}\,\lambda'\,, \qquad z' = aw'\,, \tag{3.21}$$

with

$$\lambda' = nt + \varepsilon'\,, \tag{3.22}$$

results in new equations

$$\ddot{p}\,' + 2\,\mathrm{i}\, n\dot{p}\,' - \frac{3}{2} n^2 (p\,' + q') = n^2 P\,'\,, \tag{3.23}$$

$$\ddot{w}' + n^2 w' = n^2 W' \tag{3.24}$$

with the right–hand members

$$P' = -\frac{3}{2}(p'+q') + \left(\frac{a^3}{r'^3} - 1\right)(1-p') - \frac{1}{n^2 a}(X' + \mathrm{i}\,Y')\exp(-\mathrm{i}\,\lambda'), \tag{3.25}$$

$$W' = -\left(\frac{a^3}{r'^3} - 1\right)w' + \frac{1}{n^2 a}Z'\,. \tag{3.26}$$

The mean motion n is always regarded as an observed quantity, its value being independent of the perturbations taken into account. On the contrary, $\varepsilon' \neq \varepsilon$ in the general case. The functions p', w' and $\varepsilon'$are considered as the new variables. Putting

$$p' = p + \delta p\,, \quad w' = w + \delta w\,, \quad \varepsilon' = \varepsilon + \delta\varepsilon\,, \quad \lambda' = \lambda + \delta\lambda\,, \quad \delta\lambda = \delta\varepsilon\,, \tag{3.27}$$

one gets the equations to determine δp and δw

$$\delta\ddot{p} + 2\,\mathrm{i}\,n\delta\dot{p} - \frac{3}{2}n^2(\delta p + \delta q) = n^2\delta P\,, \tag{3.28}$$

$$\delta\ddot{w} + n^2\delta w = n^2\delta W \tag{3.29}$$

with the right–hand members

$$\delta P = P' - P\,, \qquad \delta W = W' - W\,. \tag{3.30}$$

The value of $\delta\varepsilon$ remains arbitrary so far. To begin iterations, the right–hand member δP is to be computed with $\delta\varepsilon = 0$. Then the quantity $\delta\varepsilon$ is chosen to provide vanishing of the imaginary part of the constant term of this right–hand member i.e.

$$\Im(\delta P^*) = 0\,. \tag{3.31}$$

As seen from the structure of the right–hand member it is always possible. Indeed, from (3.2) and (3.21) there results

$$r^2 = a^2\left[(1-p)(1-q) + w^2\right], \tag{3.32}$$

$$r'^2 = a^2\left[(1-p')(1-q') + w'^2\right]. \tag{3.33}$$

It is easy to find

$$\left(\frac{r'}{a}\right)^2 = \left(\frac{r}{a}\right)^2(1-S)\,,$$

$$S = \left(\frac{a}{r}\right)^2\left[(1-q)\delta p + (1-p)\delta q - 2w\delta w - \delta p\delta q - (\delta w)^2\right] \tag{3.34}$$

and

$$\left(\frac{a}{r'}\right)^3 = \left(\frac{a}{r}\right)^3(1+T)\,, \qquad T = (1-S)^{-\frac{3}{2}} - 1 = \sum_{k=1}^{\infty}\frac{\left(\frac{3}{2}\right)_k}{(1)_k}S^k\,. \tag{3.35}$$

One has therefore

$$\delta P = -\frac{3}{2}(\delta p + \delta q) + (1-p)\left(\frac{a}{r}\right)^3 T - \left[\left(\frac{a}{r}\right)^3 (1+T) - 1\right]\delta p -$$
$$- \frac{1}{n^2 a}(X + \mathrm{i}\, Y)\exp(-\mathrm{i}\,\lambda)[\exp(-\mathrm{i}\,\delta\lambda) - 1] -$$
$$- \frac{1}{n^2 a}(\delta X + \mathrm{i}\,\delta Y)\exp(-\mathrm{i}\,\lambda'), \qquad (3.36)$$

$$\delta W = -w\left(\frac{a}{r}\right)^3 T - \left[\left(\frac{a}{r}\right)^3 (1+T) - 1\right]\delta w + \frac{1}{n^2 a}\delta Z\,. \qquad (3.37)$$

Thus, if the condition (3.31) is not fulfilled under $\delta\varepsilon = 0$ then one should determine $\delta\varepsilon$ from (3.36) to satisfy this condition. At the first step of iterations

$$\delta P = -\frac{1}{n^2 a}(\delta X + \mathrm{i}\,\delta Y)\exp(-\mathrm{i}\,\lambda) + \frac{1}{n^2 a}(X + \mathrm{i}\,Y)\exp(-\mathrm{i}\,\lambda)\,\mathrm{i}\,\delta\varepsilon\,. \qquad (3.38)$$

Therefore, the initial value for $\delta\varepsilon$ should be

$$\delta\varepsilon = \frac{-n^2 a\,\Im(P^*) + \Im\big\{(\delta X + \mathrm{i}\,\delta Y)\exp(-\mathrm{i}\,\lambda)\big\}^*}{\Re\big\{(X + \mathrm{i}\,Y)\exp(-\mathrm{i}\,\lambda)\big\}^*}\,, \qquad (3.39)$$

the asterisk denoting the constant part of the quantity in braces. The iterations exposed above are applied to the equations (3.28) and (3.29). In the process of iterations the value of $\delta\varepsilon$ may be improved, if necessary.The first term in the numerator of (3.39) is not needed if the initial approximate solution of the equations (3.4) and (3.5) satisfies the condition (3.18). On the other hand, the correction (3.39) with $\delta X = \delta Y = 0$ provides the validity of the condition (3.18) for the initial solution.

In performing the iterations of the present section to determine the relativistic planetary perturbations within the VSOP solution one has to deal with polynomial–exponential series of the type

$$S = \sum_k S_k(t)\exp\mathrm{i}(k\lambda)\,, \qquad (3.40)$$

where k is a multi–index $k = (k_1, \dots, k_N)$, λ being a vector with components $\lambda_1, \dots, \lambda_N$ and $(k\lambda) = k_1\lambda_1 + \dots + k_N\lambda_N$. The coefficients $S_k(t)$ are polynomials in t with complex coefficients. The polynomial structure of these coefficients is related to the secular motions of the perihelia and nodes of the planetary orbits. The further sections of the present paper deal with the problem of the relativistic planetary perturbations within the framework of the general planetary theory.

4. **Relativistic general planetary theory**

The initial variables of the general planetary theory (GPT) (Brumberg, 1995) result from the transformation of the planetary heliocentric rectangular coordinates x_i, y_i, z_i and velocity components $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ as follows:

$$x_i + \mathrm{i}\, y_i = a_i(1-p_i)\exp\mathrm{i}\,\lambda_i\,, \qquad z_i = a_i w_i\,, \tag{4.1}$$

$$\dot{x}_i + \mathrm{i}\,\dot{y}_i = a_i\left[-\dot{p}_i + \mathrm{i}\, n_i(1-p_i)\right]\exp\mathrm{i}\,\lambda_i\,, \qquad \dot{z}_i = a_i\dot{w}_i\,, \tag{4.2}$$

with

$$\lambda_i = n_i t + \varepsilon_i\,, \qquad n_i^2 a_i^3 = G(M_0 + M_i)\,. \tag{4.3}$$

The new equations of motion are

$$\ddot{p}_i + 2\,\mathrm{i}\, n_i\dot{p}_i - \frac{3}{2}n_i^2(p_i+q_i) = n_i^2 P_i\,, \tag{4.4}$$

$$\ddot{w}_i + n_i^2 w_i = n_i^2 W_i\,. \tag{4.5}$$

The right–hand members P_i and W_i are related to the initial components X_i, Y_i, Z_i of the disturbing accelerations $\mathbf{F}_N^{(i)} + \mathbf{F}_S^{(i)} + \mathbf{F}_R^{(i)}$ by means of (3.6) and (3.7) with attributing index i to all variables. The first step in GPT is to find an intermediary, i.e. a particular planar quasi–periodic solution

$$p_i = p_i^{(0)}\,, \qquad w_i = 0\,, \tag{4.6}$$

not dependent on the eccentricities and inclinations of the planetary orbits. This solution is presented by the trigonometric series in multiples of the differences of the mean longitudes λ_i with the coefficients dependent on the mean motions n_i and the major semi–axes a_i and expanded in powers of the small parameters μ and σ. Putting

$$p_i = p_i^{(0)} + \delta p_i \tag{4.7}$$

and separating explicitly the terms linear in the eccentricities and inclinations one has

$$P_i = P_i^{(0)} - \sum_{j=1}^{N}\left(K_{ij}\delta p_j + L_{ij}\delta q_j - \frac{1}{\mathrm{i}\, n_j}K'_{ij}\delta\dot{p}_j + \frac{1}{\mathrm{i}\, n_j}L'_{ij}\delta\dot{q}_j\right) + P_i^* \tag{4.8}$$

and

$$W_i = -\sum_{j=1}^{N}\left(M_{ij}w_j - \frac{1}{\mathrm{i}\, n_j}M'_{ij}\dot{w}_j\right) + W_i^*\,, \tag{4.9}$$

N still standing for the number of the planets. The coefficients K_{ij}, L_{ij}, M_{ij}, K'_{ij}, L'_{ij}, M'_{ij} are functions of intermediary starting with the first order terms with respect to the parameters μ and σ. The functions P_i^* and W_i^* are at least of the second order of smallness with respect to the eccentricities and inclinations. The new equations of motion may be described in the form

$$\delta\ddot{p}_i + 2\,\mathrm{i}\,n_i\delta\dot{p}_i + n_i^2\sum_{j=1}^{N}\Bigg[\Big(-\frac{3}{2}\delta_{ij}+K_{ij}\Big)\delta p_j + \Big(-\frac{3}{2}\delta_{ij}+L_{ij}\Big)\delta q_j - \\ - \frac{1}{\mathrm{i}\,n_j}K'_{ij}\delta\dot{p}_j + \frac{1}{\mathrm{i}\,n_j}L'_{ij}\delta\dot{q}_j\Bigg] = n_i^2 P_i^* \tag{4.10}$$

and

$$\ddot{w}_i + n_i^2\sum_{j=1}^{N}\Bigg[\Big(\delta_{ij}+M_{ij}\Big)w_j - \frac{1}{\mathrm{i}\,n_j}M'_{ij}\dot{w}_j\Bigg] = n_i^2 W_i^* \tag{4.11}$$

or in the matrix form

$$\dot{V} = AV + B \tag{4.12}$$

with block matrices

$$V = \begin{pmatrix} V' \\ V'' \end{pmatrix}, \qquad A = \begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix}, \qquad B = \begin{pmatrix} B' \\ B'' \end{pmatrix}, \tag{4.13}$$

where

$$V' = \begin{pmatrix} \delta p \\ \delta q \\ \delta\dot{p} \\ \delta\dot{q} \end{pmatrix}, \quad V'' = \begin{pmatrix} w \\ \dot{w} \end{pmatrix}, \quad B' = \begin{pmatrix} 0 \\ 0 \\ \mathcal{N}^2 P^* \\ \mathcal{N}^2 \bar{P}^* \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 \\ \mathcal{N}^2 W^* \end{pmatrix} \tag{4.14}$$

and

$$A'' = \begin{pmatrix} 0 & E \\ -\mathcal{N}^2(E+M) & -\,\mathrm{i}\,\mathcal{N}^2 M'\mathcal{N}^{-1} \end{pmatrix}. \tag{4.15}$$

As for 4×4 block matrix A' is concerned, it contains 6 zero and 2 diagonal unit blocks

$$A_{11} = A_{12} = A_{14} = A_{21} = A_{22} = A_{23} = 0\,, \quad A_{13} = A_{24} = E$$

as well as following blocks in the third line

$$A_{31} = \mathcal{N}^2\left(\frac{3}{2}E - K\right), \qquad A_{33} = -\,\mathrm{i}\,\mathcal{N}\left(2E + \mathcal{N}K'\mathcal{N}^{-1}\right), \\ A_{32} = \mathcal{N}^2\left(\frac{3}{2}E - L\right), \qquad A_{34} = \mathrm{i}\,\mathcal{N}^2 L'\mathcal{N}^{-1}\,. \tag{4.16}$$

The elements of the fourth line are obtained from these elements by complex conjugation

$$A_{41} = \bar{A}_{32}\,, \quad A_{42} = \bar{A}_{31}\,, \quad A_{43} = \bar{A}_{34}\,, \quad A_{44} = \bar{A}_{33}\,.$$

Here E stands for $N\times N$ unit matrix, $\mathcal{N}$ is $N\times N$ diagonal matrix of the mean motions, δp, δq, w, $\delta\dot p$, $\delta\dot q$, $\dot w$, P^* and W^* are N–vectors of the corresponding variables and right–hand members, the bar denotes as before a complex conjugate quantity. K, L, M, K', L' and M' are $N\times N$ matrices of the coefficients K_{ij}, L_{ij}, M_{ij}, K'_{ij}, L'_{ij} and M'_{ij}, respectively, with $\bar M = M$ and $\bar M' = -M'$.

In accordance with the GPT technique the transformation

$$V = JX\,,\qquad X = J^{-1}V\,, \tag{4.17}$$

with the matrix J of the same block structure as A

$$J' = \begin{pmatrix} E & -\frac{2}{3}E & -\frac{1}{2}E & \frac{3}{2}E \\ -E & -\frac{2}{3}E & \frac{3}{2}E & -\frac{1}{2}E \\ 0 & \mathrm{i}\mathcal{N} & -\frac{\mathrm{i}}{2}\mathcal{N} & -\frac{3\mathrm{i}}{2}\mathcal{N} \\ 0 & -\mathrm{i}\mathcal{N} & \frac{3\mathrm{i}}{2}\mathcal{N} & \frac{\mathrm{i}}{2}\mathcal{N} \end{pmatrix}, \qquad J'' = \begin{pmatrix} E & E \\ \mathrm{i}\mathcal{N} & -\mathrm{i}\mathcal{N} \end{pmatrix}, \tag{4.18}$$

$$J'^{-1} = \begin{pmatrix} \frac{1}{2}E & -\frac{1}{2}E & -\mathrm{i}\mathcal{N}^{-1} & -\mathrm{i}\mathcal{N}^{-1} \\ -3E & -3E & \frac{3\mathrm{i}}{2}\mathcal{N}^{-1} & -\frac{3\mathrm{i}}{2}\mathcal{N}^{-1} \\ -\frac{3}{2}E & -\frac{3}{2}E & \frac{\mathrm{i}}{2}\mathcal{N}^{-1} & -\frac{3\mathrm{i}}{2}\mathcal{N}^{-1} \\ -\frac{3}{2}E & -\frac{3}{2}E & \frac{3\mathrm{i}}{2}\mathcal{N}^{-1} & -\frac{\mathrm{i}}{2}\mathcal{N}^{-1} \end{pmatrix}, \; J''^{-1} = \begin{pmatrix} \frac{1}{2}E & -\frac{\mathrm{i}}{2}\mathcal{N}^{-1} \\ \frac{1}{2}E & \frac{\mathrm{i}}{2}\mathcal{N}^{-1} \end{pmatrix} \tag{4.19}$$

and new variables

$$X = \begin{pmatrix} \xi \\ \eta \\ u \\ \bar u \\ v \\ \bar v \end{pmatrix}, \qquad \bar\xi = -\xi\,, \qquad \bar\eta = \eta\,, \tag{4.20}$$

reduces the system (4.12) to the quasi–Jordan form

$$\dot X = \mathrm{i}\mathcal{N}\,\left[(\mathcal{P}+Q)X + R\right] \tag{4.21}$$

with a constant Jordan matrix

$$\mathcal{P} = \begin{pmatrix} 0 & E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & -E \end{pmatrix}, \tag{4.22}$$

the right–hand members

$$R = -\mathrm{i}\mathcal{N}^{-1}J^{-1}B \tag{4.23}$$

and quasi–periodic matrix

$$Q = -\,\mathrm{i}\,\mathcal{N}^{-1}J^{-1}AJ - \mathcal{P}\,. \tag{4.24}$$

The diagonal matrix $\mathcal{N}$ in (4.21), (4.23) and (4.24) is to be treated as a block–factor. The quantities $X = \|X_\kappa\|$ and $R = \|R_\kappa\|$ ($\kappa = 1, 2, \ldots, 6$) represent the block vectors where each block is a N–vector of the corresponding variables

$$X_1 = \xi\,, \qquad X_2 = \eta\,, \qquad X_3 = u\,, \qquad X_5 = v \tag{4.25}$$

and

$$\begin{aligned} R_1 &= -(P^* + \bar{P}^*)\,, \qquad & R_3 &= \tfrac{1}{2}(P^* - 3\bar{P}^*)\,, \\ R_2 &= \tfrac{3}{2}(P^* - \bar{P}^*)\,, \qquad & R_5 &= -\tfrac{1}{2}W^*\,. \end{aligned} \tag{4.26}$$

One has therewith

$$\bar{X}_1 = -X_1, \qquad \bar{X}_2 = X_2, \qquad X_4 = \bar{X}_3, \qquad X_6 = \bar{X}_5 \tag{4.27}$$

and

$$\bar{R}_1 = R_1, \qquad \bar{R}_2 = -R_2, \qquad R_4 = -\bar{R}_3, \qquad R_6 = -\bar{R}_5\,. \tag{4.28}$$

The matrices $\mathcal{P} = \|\mathcal{P}_{\kappa\nu}\|$ and $Q = \|Q_{\kappa\nu}\|$ ($\kappa,\, \nu = 1, 2, \ldots, 6$) are composed by square blocks, all of which representing $N \times N$ matrix. $\mathcal{P}$ is a constant Jordan matrix with non–zero blocks

$$\mathcal{P}_{12} = \mathcal{P}_{33} = \mathcal{P}_{55} = E\,, \qquad \mathcal{P}_{44} = \mathcal{P}_{66} = -E\,. \tag{4.29}$$

Q is a quasi–periodic matrix consisting of 36 square blocks. 16 blocks with indices $(\kappa, 5)$, $(\kappa, 6)$, $(5, \kappa)$ and$(6, \kappa)$ for $\kappa = 1, 2, 3, 4$ are zero blocks. 12 essential blocks have the values

$$\begin{aligned}
Q_{11} &= K - L - \bar{K} + \bar{L}\,, \\
Q_{12} &= -\tfrac{2}{3}(K + L + \bar{K} + \bar{L}) - (K' + L' + \bar{K}' + \bar{L}')\,, \\
Q_{21} &= \tfrac{3}{2}(-K + L - \bar{K} + \bar{L}), \\
Q_{22} &= K + L - \bar{K} - \bar{L} + \tfrac{3}{2}(K' + L' - \bar{K}' - \bar{L}')\,, \\
Q_{13} &= \tfrac{1}{2}(-K + 3L + 3\bar{K} - \bar{L}) + \tfrac{1}{2}(K' + 3L' + 3\bar{K}' + \bar{L}')\,, \\
Q_{23} &= \tfrac{3}{4}(K - 3L + 3\bar{K} - \bar{L}) + \tfrac{3}{4}(-K' - 3L' + 3\bar{K}' + \bar{L}')\,, \\
Q_{31} &= \tfrac{1}{2}(-K + L - 3\bar{K} + 3\bar{L})\,, \\
Q_{32} &= \tfrac{1}{3}(K + L - 3\bar{K} - 3\bar{L}) + \tfrac{1}{2}(K' + L' - 3\bar{K}' - 3\bar{L}')\,, \\
Q_{33} &= \tfrac{1}{4}(K - 3L + 9\bar{K} - 3\bar{L}) + \tfrac{1}{4}(-K' - 3L' + 9\bar{K}' + 3\bar{L}')\,, \\
Q_{34} &= \tfrac{1}{4}(-3K + L - 3\bar{K} + 9\bar{L}) + \tfrac{1}{4}(-3K' - L' + 3\bar{K}' + 9\bar{L}')\,, \\
Q_{55} &= \tfrac{1}{2}(M - M')\,, \\
Q_{56} &= \tfrac{1}{2}(M + M')\,.
\end{aligned} \tag{4.30}$$

The remaining 8 blocks satisfy the conjugation conditions

$$
\begin{aligned}
&Q_{14} = \bar{Q}_{13}\,, \quad Q_{24} = -\bar{Q}_{23}\,, \quad Q_{41} = \bar{Q}_{31}\,, \quad Q_{42} = -\bar{Q}_{32}\,,\\
&Q_{43} = -\bar{Q}_{34}\,, \quad Q_{44} = -\bar{Q}_{33}\,, \quad Q_{65} = -\bar{Q}_{56}\,, \quad Q_{66} = -\bar{Q}_{55}\,.
\end{aligned}
\tag{4.31}
$$

The equations (4.21) have the same form as in the Newtonian GPT and all subsequent steps may be performed just as in the Newtonian case (Brumberg, 1995). For our purposes it is sufficient to consider only the intermediary and the secular system in the linear (with respect to the eccentricities and inclinations) approximation.

5. **Right–hand members of the equations of motion**

Originally, the right–hand members (4.4) and (4.5) are represented by the series

$$
P_i = \underset{00}{P}_i + \mu \underset{10}{P}_i + \sigma \underset{01}{P}_i + \mu\sigma \underset{11}{P}_i + \ldots \tag{5.1}
$$

and

$$
W_i = \underset{00}{W}_i + \mu \underset{10}{W}_i + \sigma \underset{01}{W}_i + \mu\sigma \underset{11}{W}_i + \ldots\ . \tag{5.2}
$$

The free terms and the coefficients in μ of these series are given in different forms in (Brumberg. 1995). The Schwarzschild and relativistic planetary terms are

$$
\underset{01}{P}_i = -\frac{A}{n_i^2}\left[A'_{i0}(1-p_i) + \mathrm{i}\, n_i B'_{i0}\left(1 - p_i - \frac{\dot{p}_i}{\mathrm{i}\, n_i}\right)\right], \tag{5.3}
$$

$$
\underset{01}{W}_i = \frac{A}{n_i^2}\,(A'_{i0} w_i + B'_{i0}\dot{w}_i)\,, \tag{5.4}
$$

$$
\begin{aligned}
\underset{11}{P}_i = -\,\frac{A}{n_i^2}\Bigg\{&(A^*_{i0} + \kappa_{ii}A_{0i})(1-p_i) + \mathrm{i}\, n_i\,(B^*_{i0} + \kappa_{ii}B_{0i})\left(1 - p_i - \frac{\dot{p}_i}{\mathrm{i}\, n_i}\right) +\\
&+ \sum_{j\neq i} \kappa_{jj}\Bigg[A_{ij}(1-p_i) + (A_{0j} - A_{ij})\frac{a_j}{a_i}(1-p_j)\zeta_{ij}^{-1} +\\
&+ \mathrm{i}\, n_i B_{ij}\left(1 - p_i - \frac{\dot{p}_i}{\mathrm{i}\, n_i}\right) + \mathrm{i}\, n_j (B_{0j} - B_{ij})\frac{a_j}{a_i}\left(1 - p_j - \frac{\dot{p}_j}{\mathrm{i}\, n_j}\right)\zeta_{ij}^{-1}\Bigg]\Bigg\}
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
\underset{11}{W}_i = \frac{A}{n_i^2}\Bigg\{&(A^*_{i0} + \kappa_{ii}A_{0i})w_i + (B^*_{i0} + \kappa_{ii}B_{0i})\dot{w}_i +\\
&+ \sum_{j\neq i}\kappa_{jj}\left[A_{ij}w_i + B_{ij}\dot{w}_i + (A_{0j} - A_{ij})\frac{a_j}{a_i}w_j + (B_{0j} - B_{ij})\frac{a_j}{a_i}\dot{w}_j\right]\Bigg\},
\end{aligned}
\tag{5.6}
$$

with

$$\zeta_{ij} = \exp \mathrm{i}(\lambda_i - \lambda_j)\,. \tag{5.7}$$

Separating the intermediate solution (4.6) and substituting (4.7) one gets each of the given above coefficients A and B as the sum of the part $A^{(0)}$ or $B^{(0)}$ dependent only on the intermediary and the part δA or δB dependent on the eccentricities and inclinations. This paper treats any relativistic terms only within the first power with respect to the eccentricities and inclinations. In result, the expressions (5.3)–(5.6) take the form

$$\begin{aligned}\underset{01}{P}_i = &-\frac{A}{n_i^2}\Bigg[A_{i0}^{\prime(0)}\left(1-p_i^{(0)}\right)+\mathrm{i}\,n_i B_{i0}^{\prime(0)}\left(1-p_i^{(0)}-\frac{\dot p_i^{(0)}}{\mathrm{i}\,n_i}\right)-\\ &-A_{i0}^{\prime(0)}\delta p_i-\mathrm{i}\,n_i B_{i0}^{\prime(0)}\left(\delta p_i+\frac{\delta\dot p_i}{\mathrm{i}\,n_i}\right)+\left(1-p_i^{(0)}\right)\delta A_{i0}'+\\ &+\mathrm{i}\,n_i\left(1-p_i^{(0)}-\frac{\dot p_i^{(0)}}{\mathrm{i}\,n_i}\right)\delta B_{i0}'\Bigg],\end{aligned} \tag{5.8}$$

$$\underset{01}{W}_i = \frac{A}{n_i^2}\left(A_{i0}^{\prime(0)}w_i + B_{i0}^{\prime(0)}\dot w_i\right). \tag{5.9}$$

$$\begin{aligned}\underset{11}{P}_i = &-\frac{A}{n_i^2}\Bigg\{A_{i0}^{*(0)}+\kappa_{ii}A_{0i}^{(0)}+\mathrm{i}\,n_i\left(B_{i0}^{*(0)}+\kappa_{ii}B_{0i}^{(0)}\right)-\\ &-\left(A_{i0}^{*(0)}+\kappa_{ii}A_{0i}^{(0)}\right)\delta p_i-\mathrm{i}\,n_i\left(B_{i0}^{*(0)}+\kappa_{ii}B_{0i}^{(0)}\right)\left(\delta p_i+\frac{\delta\dot p_i}{\mathrm{i}\,n_i}\right)+\\ &+\delta A_{i0}^{*}+\kappa_{ii}\delta A_{0i}+\mathrm{i}\,n_i\left(\delta B_{i0}^{*}+\kappa_{ii}\delta B_{0i}\right)+\\ &+\sum_{j\neq i}\kappa_{jj}\Bigg[A_{ij}^{(0)}+\mathrm{i}\,n_i B_{ij}^{(0)}+\left(A_{0j}^{(0)}-A_{ij}^{(0)}\right)\frac{a_j}{a_i}\zeta_{ij}^{-1}+\mathrm{i}\,n_j\times\\ &\times\left(B_{0j}^{(0)}-B_{ij}^{(0)}\right)\frac{a_j}{a_i}\zeta_{ij}^{-1}-A_{ij}^{(0)}\delta p_i-\mathrm{i}\,n_i B_{ij}^{(0)}\left(\delta p_i+\frac{\delta\dot p_i}{\mathrm{i}\,n_i}\right)-\\ &-\left(A_{0j}^{(0)}-A_{ij}^{(0)}\right)\frac{a_j}{a_i}\zeta_{ij}^{-1}\delta p_j-\mathrm{i}\,n_j\left(B_{0j}^{(0)}-B_{ij}^{(0)}\right)\frac{a_j}{a_i}\zeta_{ij}^{-1}\left(\delta p_j+\frac{\delta\dot p_j}{\mathrm{i}\,n_j}\right)+\\ &+\delta A_{ij}+\mathrm{i}\,n_i\delta B_{ij}+\frac{a_j}{a_i}\zeta_{ij}^{-1}\left(\delta A_{0j}-\delta A_{ij}\right)+\mathrm{i}\,n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\left(\delta B_{0j}-\delta B_{ij}\right)\Bigg]\Bigg\},\end{aligned} \tag{5.10}$$

$$\begin{aligned}\underset{11}{W}_i = &\frac{A}{n_i^2}\Bigg\{\left(A_{i0}^{*(0)}+\kappa_{ii}A_{0i}^{(0)}\right)w_i+\left(B_{i0}^{*(0)}+\kappa_{ii}B_{0i}^{(0)}\right)\dot w_i+\\ &+\sum_{j\neq i}\kappa_{jj}\left[A_{ij}^{(0)}w_i+B_{ij}^{(0)}\dot w_i+\left(A_{0j}^{(0)}-A_{ij}^{(0)}\right)\frac{a_j}{a_i}w_j+\left(B_{0j}^{(0)}-B_{ij}^{(0)}\right)\frac{a_j}{a_i}\dot w_j\right]\Bigg\}.\end{aligned} \tag{5.11}$$

All the coefficients in the two last expressions (5.10) and (5.11) are needed only in the circular motion approximation ($p_i = q_i = w_i = 0$).

Within the demanded accuracy one gets

$$A'^{(0)}_{i0} = \frac{n_i^2}{a_i}\left(3 + \frac{15}{2}p_i^{(0)} + \frac{15}{2}q_i^{(0)} + \frac{\dot p_i^{(0)}}{\mathrm{i}\,n_i} - \frac{\dot q_i^{(0)}}{\mathrm{i}\,n_i}\right), \tag{5.12}$$

$$\begin{aligned}\delta A'_{i0} =& \frac{n_i^2}{a_i}\left[\left(\frac{15}{2} + \frac{93}{4}p_i^{(0)} + \frac{63}{4}q_i^{(0)} + \frac{3}{2}\frac{\dot p_i^{(0)}}{\mathrm{i}\,n_i} - \frac{1}{2}\frac{\dot q_i^{(0)}}{\mathrm{i}\,n_i}\right)\delta p_i + \right.\\ &+ \left(\frac{15}{2} + \frac{63}{4}p_i^{(0)} + \frac{93}{4}q_i^{(0)} + \frac{1}{2}\frac{\dot p_i^{(0)}}{\mathrm{i}\,n_i} - \frac{3}{2}\frac{\dot q_i^{(0)}}{\mathrm{i}\,n_i}\right)\delta q_i + \\ &+ \left(1 + \frac{3}{2}p_i^{(0)} + \frac{1}{2}q_i^{(0)} + \frac{\dot q_i^{(0)}}{\mathrm{i}\,n_i}\right)\frac{\delta \dot p_i^{(0)}}{\mathrm{i}\,n_i} - \\ &\left. - \left(1 + \frac{1}{2}p_i^{(0)} + \frac{3}{2}q_i^{(0)} - \frac{\dot p_i^{(0)}}{\mathrm{i}\,n_i}\right)\frac{\delta \dot q_i^{(0)}}{\mathrm{i}\,n_i}\right],\end{aligned} \tag{5.13}$$

$$B'^{(0)}_{i0} = -\frac{2}{a_i}\left(\dot p_i^{(0)} + \dot q_i^{(0)}\right), \tag{5.14}$$

$$\begin{aligned}\delta B'_{i0} =& -\frac{2}{a_i}\left[\frac{1}{2}\left(3\dot p_i^{(0)} + \dot q_i^{(0)}\right)\delta p_i + \frac{1}{2}\left(\dot p_i^{(0)} + 3\dot q_i^{(0)}\right)\delta q_i + \right.\\ &\left. + \left(1 + \frac{3}{2}p_i^{(0)} + \frac{1}{2}q_i^{(0)}\right)\delta \dot p_i + \left(1 + \frac{1}{2}p_i^{(0)} + \frac{3}{2}q_i^{(0)}\right)\delta \dot q_i\right].\end{aligned} \tag{5.15}$$

As mentioned above, all other coefficients are needed only within the circular motion approximation. Besides, one may make no difference between κ_{ij} and κ_{jj} and apply the third Kepler law in its simplified form $n_i^2 a_i^3 = n_j^2 a_j^3$. Taking all this into account, one finds

$$\begin{aligned}A^{*(0)}_{i0} =& -\frac{n_i^2}{a_i}\kappa_{ii} + n_i^2 \sum_{j\neq i}\kappa_{jj}\left[\frac{1}{a_j} + \frac{1}{4}\frac{a_i}{a_j^2}\left(1 - 4\frac{n_i}{n_j}\right)(\zeta_{ij} + \zeta_{ij}^{-1}) + \right.\\ &\left. + \frac{7}{2}\frac{a_i^2}{a_j^3} + \frac{4}{\Delta_{ij}} - \frac{7}{2}\left(\frac{a_i}{a_j}\right)^3 \frac{1}{\Delta_{ij}}\right],\end{aligned} \tag{5.16}$$

$$A^{(0)}_{0i} = 3\,\frac{n_i^2}{a_i}, \tag{5.17}$$

$$\begin{aligned}A^{(0)}_{ij} =& n_j^2\left(\frac{4}{\Delta_{ij}} - \frac{7}{2a_i}\right) + \frac{1}{\Delta_{ij}^3}\left[3n_i^2 a_i^2 - n_j^2\left(\frac{1}{2}a_i^2 + a_j^2\right) + a_i a_j n_j \times \right.\\ &\left. \times \left(2n_i + \frac{1}{4}n_j\right)(\zeta_{ij} + \zeta_{ij}^{-1}) - \frac{3}{8}\frac{a_i^2 a_j^2 n_j^2}{\Delta_{ij}^2}\left(\zeta_{ij}^2 + \zeta_{ij}^{-2} - 2\right)\right],\end{aligned} \tag{5.18}$$

$$B_{i0}^{*(0)} = \frac{\mathrm{i}}{2a_i^2}\sum_{j\neq i}\kappa_{jj}n_ja_j\left(\zeta_{ij}-\zeta_{ij}^{-1}\right), \tag{5.19}$$

$$B_{0i}^{(0)} = 0\,, \tag{5.20}$$

$$B_{ij}^{(0)} = \frac{\mathrm{i}\,a_ia_j}{\Delta_{ij}^3}\left(\frac{3}{2}n_j - 2n_i\right)\left(\zeta_{ij}-\zeta_{ij}^{-1}\right), \tag{5.21}$$

with Δ_{ij} denoting the mutual distance between planets i and j (for complanar circular orbits). The terms of the first power in eccentricities and inclinations are

$$\delta A_{0i} = \frac{n_i^2}{a_i}\left(9\,\delta p_i + 9\,\delta q_i + 2\,\frac{\delta\dot p_i}{\mathrm{i}\,n_i} - 2\,\frac{\delta\dot q_i}{\mathrm{i}\,n_i}\right), \qquad \delta B_{0i} = -\,\mathrm{i}\,\frac{3}{2}\frac{n_i}{a_i}\left(\frac{\delta\dot p_i}{\mathrm{i}\,n_i} + \frac{\delta\dot q_i}{\mathrm{i}\,n_i}\right), \tag{5.22}$$

$$\begin{aligned}\delta A_{i0}^* =& A_{i001}\,\delta p_i + A_{i002}\,\delta q_i + A_{i003}\,\frac{\delta\dot p_i}{\mathrm{i}\,n_i} + A_{i004}\,\frac{\delta\dot q_i}{\mathrm{i}\,n_i} + \\ &+ \sum_{j\neq i}\left(A_{i0j1}\,\delta p_j + A_{i0j2}\,\delta q_j + A_{i0j3}\,\frac{\delta\dot p_j}{\mathrm{i}\,n_j} + A_{i0j4}\,\frac{\delta\dot q_j}{\mathrm{i}\,n_j}\right),\end{aligned} \tag{5.23}$$

$$\begin{aligned}\delta A_{ij} =& A_{ij01}\,\delta p_i + A_{ij02}\,\delta q_i + A_{ij03}\,\frac{\delta\dot p_i}{\mathrm{i}\,n_i} + A_{ij04}\,\frac{\delta\dot q_i}{\mathrm{i}\,n_i} + \\ &+ A_{ijj1}\,\delta p_j + A_{ijj2}\,\delta q_j + A_{ijj3}\,\frac{\delta\dot p_j}{\mathrm{i}\,n_j} + A_{ijj4}\,\frac{\delta\dot q_j}{\mathrm{i}\,n_j}\end{aligned} \tag{5.24}$$

with similar expressions for δB_{i0}^* and δB_{ij}. The coefficients of these expansions resulting from the general expressions for A_{i0}^*, B_{i0}^*, A_{ij} and B_{ij} are given in the Appendix 1.

6. **Construction of the intermediate solution**

At each step of constructing the intermediate solution the right–hand member of equation (4.4) is presented by a series

$$P_i^{(0)} = \sum P_k^{(i)}\exp\mathrm{i}(k\lambda)\,, \tag{6.1}$$

k being a N multi–index with

$$(k\lambda) = k_1\lambda_1 + \ldots + k_N\lambda_N\,, \qquad k_1 + k_2 + \ldots + k_N = 0\,. \tag{6.2}$$

After integration the intermediate solution is presented in the same form with the coefficients

$$p_0^{(i)} = -\frac{1}{3}P_0^{(i)}\,, \qquad p_k^{(i)} = n_i^2\,\frac{\left[(kn)^2 - 2n_i(kn) + \frac{3}{2}n_i^2\right]P_k^{(i)} - \frac{3}{2}n_i^2P_{-k}^{(i)}}{(kn)^2\left[n_i^2 - (kn)^2\right]}\,. \tag{6.3}$$

The intermediate solution is expanded therewith in powers of μ and σ

$$p_i^{(0)} = \mu \underset{10}{p}_i + \sigma \underset{01}{p}_i + \mu\sigma \underset{11}{p}_i + \dots \,. \tag{6.4}$$

From (5.8), (5.12) and (6.3) it is seen that within the first order in σ the only relativistic contribution into the intermediate solution is presented by the constant term

$$\underset{01}{p}_i = \frac{A}{a_i} \,. \tag{6.5}$$

The intermediary solution (6.4) with (6.5) is substituted into the right–hand members (5.1) and (5.2) to re–order terms in powers of μ and σ. More specifically, the right–hand members of the equations for the intermediary are presented in the form

$$P_i^{(0)} = \mu P_{10}^{(i)} + \sigma P_{01}^{(i)} + \mu\sigma P_{11}^{(i)} + \dots \,, \tag{6.6}$$

$$P_{10}^{(i)} = \sum_{j \neq i} \kappa_{ij} \psi_{000000}^{(ij)} \,, \tag{6.7}$$

$$P_{01}^{(i)} = -3\frac{A}{a_i} \,, \tag{6.8}$$

$$\begin{aligned} P_{11}^{(i)} =& \frac{A}{a_i}\Bigg[\sum_{j\neq i}\kappa_{ij}\Big(\psi_{100000}^{(ij)} + \psi_{010000}^{(ij)} + \frac{a_i}{a_j}\psi_{001000}^{(ij)} + \frac{a_i}{a_j}\psi_{000100}^{(ij)}\Big) - \\ & - 3\underset{10}{p}_i - 3\underset{10}{q}_i - \frac{3}{\mathrm{i}\, n_i}\underset{10}{\dot{p}}_i - \frac{1}{\mathrm{i}\, n_i}\underset{10}{\dot{q}}_i\Bigg] - \frac{A}{n_i^2}\Bigg\{A_{i0}^{*(0)} + \kappa_{ii}A_{0i}^{(0)} + \\ & + \mathrm{i}\, n_i\Big(B_{i0}^{*(0)} + \kappa_{ii}B_{0i}^{(0)}\Big) + \sum_{j\neq i}\kappa_{jj}\Bigg[A_{ij}^{(0)} + \mathrm{i}\, n_i B_{ij}^{(0)} + \\ & + \Big(A_{0j}^{(0)} - A_{ij}^{(0)}\Big)\frac{a_j}{a_i}\zeta_{ij}^{-1} + \mathrm{i}\, n_j\Big(B_{0j}^{(0)} - B_{ij}^{(0)}\Big)\frac{a_j}{a_i}\zeta_{ij}^{-1}\Bigg]\Bigg\} \end{aligned} \tag{6.9}$$

with values (5.16)–(5.21). Again one may make no difference here between κ_{ij} and κ_{jj}. Various expressions for occurring here functions $\psi_{klrsmt}^{(ij)}$ are given in (Brumberg, 1995). These functions are expanded in the Fourier series

$$\psi_{klrsmt}^{(ij)} = \sum_{\sigma=-\infty}^{\infty} \psi_\sigma^{(ij;klrsmt)}\zeta_{ij}^\sigma \,, \tag{6.10}$$

the coefficients may be easily computed by means of the hypergeometric series. These coefficients may be also expressed in terms of the Laplace symmetric coefficients $c_n^{(\sigma)}(a_i, a_j)$, defined as the Fourier coefficients of the expansion

$$\Delta_{ij}^{-n} = \frac{1}{2}\sum_{\sigma=-\infty}^{\infty} c_n^{(\sigma)}(a_i, a_j)\zeta_{ij}^\sigma \,, \qquad c_n^{(-\sigma)} = c_n^{(\sigma)} \tag{6.11}$$

and represented, e.g., with the aid of the hypergeometric functions

$$\frac{1}{2}c_n^{(\sigma)}(a,a') = \frac{\left(\frac{n}{2}\right)_{|\sigma|}}{(1)_{|\sigma|}(aa')^{\frac{n}{2}}}\,\alpha^{|\sigma|+\frac{n}{2}}\,F\Big(\tfrac{n}{2}\,,\,\tfrac{n}{2}+|\sigma|\,,1+|\sigma|\,,\alpha^2\Big)\,, \tag{6.12}$$

with

$$\alpha = \frac{\min\{a,a'\}}{\max\{a,a'\}}\,. \tag{6.13}$$

In particular,

$$p_i = \sum_{j\neq i}\kappa_{ij}\sum_{\sigma=-\infty}^{\infty}p_\sigma^{(ij)}\zeta_{ij}^\sigma\,, \tag{6.14}$$

with coefficients

$$p_0^{(ij)} = -\frac{1}{3}\psi_0^{(ij;000000)}\,, \tag{6.15}$$

$$p_\sigma^{(ij)} = \frac{m_{ij}^2}{\sigma^2(m_{ij}^2-\sigma^2)}\left[\left(\sigma^2-2\sigma m_{ij}+\frac{3}{2}m_{ij}^2\right)\psi_\sigma^{(ij;000000)}-\frac{3}{2}m_{ij}^2\psi_{-\sigma}^{(ij;000000)}\right], \tag{6.16}$$

with

$$m_{ij} = \frac{n_i}{n_i-n_j} \tag{6.17}$$

and

$$\psi_\sigma^{(ij;000000)} = \left(\frac{a_i}{a_j}\right)^2\delta_{\sigma,-1}+\frac{1}{2}a_i^3c_3^{(\sigma)}(a_i,a_j)-\frac{1}{2}a_i^2a_jc_3^{(\sigma+1)}(a_i,a_j)\,. \tag{6.18}$$

Besides,

$$\psi_\sigma^{(ij;100000)} = \frac{1}{4}a_i^3c_3^{(\sigma)}(a_i,a_j)\,, \tag{6.19}$$

$$\psi_\sigma^{(ij;010000)} = \frac{1}{4}(3+2\sigma)a_i^3c_3^{(\sigma)}(a_i,a_j)-\frac{1}{2}(1+\sigma)a_i^2a_jc_3^{(\sigma+1)}(a_i,a_j)\,, \tag{6.20}$$

$$\psi_\sigma^{(ij;001000)} = \frac{1}{2}\left(\frac{a_i}{a_j}\right)^2\delta_{\sigma,-1}-\frac{1}{4}a_i^2a_jc_3^{(\sigma+1)}(a_i,a_j), \tag{6.21}$$

$$\psi_\sigma^{(ij;000100)} = \frac{3}{2}\left(\frac{a_i}{a_j}\right)^2\delta_{\sigma,-1}-\frac{1}{2}\sigma a_i^3c_3^{(\sigma)}(a_i,a_j)-\frac{1}{4}(1-2\sigma)a_i^2a_jc_3^{(\sigma+1)}(a_i,a_j)\,. \tag{6.22}$$

Thus all functions occurring in (6.6) become known. It enables one to find the expansion (6.1). After integrating by means of (6.3) one finds in the same form the relativistic planetary perturbations not dependent on the eccentricities and inclinations of the planetary orbits.

7. **Construction of the secular system**

In accordance with the GPT technique (Brumberg, 1955) the system (4.21) is transformed by means of

$$X = (E+S)Y+\Gamma(Y,t) \tag{7.1}$$

to the system

$$\dot{Y} = \mathrm{i}\mathcal{N}\big[HY + F(Y,t)\big] \tag{7.2}$$

with the blocks $Y_1 = Y_2 = 0$. The matrix S is found by iterations with respect to μ and σ from the system

$$G = Q(E+S) - \mathcal{N}^{-1}S\mathcal{N}G^* \,, \tag{7.3}$$

$$G = G^* + G^+ \,, \tag{7.4}$$

$$\dot{S} + \mathrm{i}(S\mathcal{N}\mathcal{P} - \mathcal{N}\mathcal{P}S) = \mathrm{i}\mathcal{N}G^+ \,, \qquad (7.5|$$

with the initial approximation $S = 0$. The splitting (7.4) is performed to ensure the integration of (7.5) without secular terms. It means that the only non–zero blocks of matrix G^* may be

$$G^*_{\kappa\nu} = \langle G_{\kappa\nu}\rangle \,, \qquad \kappa\,,\nu = 1,2 \tag{7.6}$$

and

$$\begin{aligned} G^*_{33} &= D\left\langle D^{-1}G_{33}D\right\rangle D^{-1}\,, \qquad G^*_{44} = -\bar{G}^*_{33}\,, \\ G^*_{55} &= D\left\langle D^{-1}G_{55}D\right\rangle D^{-1}\,, \qquad G^*_{66} = -\bar{G}^*_{55}\,, \end{aligned} \tag{7.7}$$

where the angle brackets means the mean value of the corresponding function and the diagonal matrix D is composed of the elements $\exp\mathrm{i}\,\lambda_i$ $(i = 1, 2, \ldots, N)$. By determining the matrices S and G one gets the matrix of the linear part of the system (7.2)

$$H = \mathcal{P} + G^* \,. \tag{7.8}$$

The quasi–periodic terms linear in the eccentricities and inclinations are determined by the matrix relations

$$\delta p = \left(-\frac{1}{2}E + c\right)Y_3 + \left(\frac{3}{2}E + d\right)\bar{Y}_3 + \ldots \,, \tag{7.9}$$

$$w = (E+f)Y_5 + (E+\bar{f})\bar{Y}_5 + \ldots \,, \tag{7.10}$$

with

$$c = S_{13} - \frac{2}{3}S_{23} - \frac{1}{2}S_{33} + \frac{3}{2}\bar{S}_{34}\,, \tag{7.11}$$

$$d = -\bar{S}_{13} - \frac{2}{3}\bar{S}_{23} + \frac{3}{2}\bar{S}_{33} - \frac{1}{2}S_{34}\,, \tag{7.12}$$

$$f = S_{55} + \bar{S}_{56}\,. \tag{7.13}$$

For the sake of completeness let us remind that the non–linear terms Γ are found by iterations from the system

$$U = R + Q\Gamma - \mathcal{N}^{-1}\Gamma_Y\mathcal{N}G^*Y - \mathcal{N}^{-1}(S+\Gamma_Y)\mathcal{N}U^* \,, \tag{7.14}$$

$$U = U^* + U^+ \,, \tag{7.15}$$

$$\Gamma_t + \mathrm{i}(\Gamma_Y\mathcal{N}\mathcal{P}Y - \mathcal{N}\mathcal{P}\Gamma) = \mathrm{i}\mathcal{N}U^+ \tag{7.16}$$

with initial value $U = R$. The splitting (7.15) is performed to ensure the integration of (7.16) without secular terms. Introducing the final set of variables Z with $Z_1 = Z_2 = 0$,

$$Y_3 = DZ_3\,, \quad Y_5 = DZ_5\,, \quad Z_4 = \bar{Z}_3\,, \quad Z_6 = \bar{Z}_5 \tag{7.17}$$

and substituting (7.17) into $U = U(Y, t)$ one gets a new function

$$V(Z, t) = U(Y, t)\,, \tag{7.18}$$

together with

$$U^*_\kappa = \langle V_\kappa(Z, t)\rangle\,, \qquad \kappa = 1, 2 \tag{7.19}$$

and

$$\begin{aligned} U^*_3 &= D\left\langle D^{-1}V_3(Z, t)\right\rangle, \qquad U^*_4 = -\bar{U}^*_3\,, \\ U^*_5 &= D\left\langle D^{-1}V_5(Z, t)\right\rangle, \qquad U^*_6 = -\bar{U}^*_4\,, \end{aligned} \tag{7.20}$$

with averaging over explicitly presented t. Having determined Γ and U one gets

$$F = U^*\,. \tag{7.21}$$

In GPT the condition $U^*_2 = 0$ is fulfilled by itself. The first two equations (7.16) for Γ_1 and Γ_2 may be integrated with total functions U_1 and U_2 in their right–hand members provided that the term $-U^*_1$ is added to Γ_2. It means, actually, that in (7.14) one may put $U^*_1 = U^*_2 = 0$. It follows from this that $F_1 = F_2 = 0$. Thus the final transformation (7.17) reduces (7.2) to the secular system

$$\begin{aligned} \dot{\alpha} &= \mathrm{i}\,\mathcal{N}[A\alpha + \Phi(\alpha, \bar{\alpha}, \beta, \bar{\beta})]\,, \\ \dot{\beta} &= \mathrm{i}\,\mathcal{N}[B\beta + \Psi(\alpha, \bar{\alpha}, \beta, \bar{\beta})] \end{aligned} \tag{7.22}$$

with $\alpha = Z_3$, $\beta = Z_5$,

$$A = \langle D^{-1}G_{33}D\rangle\,, \qquad B = \langle D^{-1}G_{55}D\rangle\,, \tag{7.23}$$

$$\Phi = \langle D^{-1}V_3(Z, t)\rangle\,, \qquad \Psi = \langle D^{-1}V_5(Z, t)\rangle\,. \tag{7.24}$$

The secular matrices are expanded in powers of μ and σ

$$\begin{aligned} A &= \mu \underset{10}{A} + \sigma \underset{01}{A} + \mu\sigma \underset{11}{A} + \ldots\,, \\ B &= \mu \underset{10}{B} + \sigma \underset{01}{B} + \mu\sigma \underset{11}{B} + \ldots\,. \end{aligned} \tag{7.25}$$

All matrices K, L, M, K', L', M' as well as the blocks of the matrix Q are expanded in the same manner. All quantities of the first power in μ known from the Newtonian GPT are given in various forms in (Brumberg, 1995). In the Schwarzschild approximation responsible for the terms of the first power in σ the matrices M and M' are zero whereas the matrices K, L, K', L' are diagonal matrices with the elements

$$\underset{01}{K}_{ii} = \underset{01}{L}_{ii} = -\underset{01}{K'}_{ii} = 3\,\underset{01}{L'}_{ii} = 3\frac{A}{a_i}\,. \tag{7.26}$$

Hence, 12 essential blocks (4.30) of the matrix Q present in the Schwarzschild approximation the diagonal matrices with the elements

$$\begin{gathered}
\underset{01}{Q}_{11}[i,i] = \underset{01}{Q}_{21}[i,i] = \underset{01}{Q}_{22}[i,i] = \underset{01}{Q}_{31}[i,i] = \underset{01}{Q}_{55}[i,i] = \underset{01}{Q}_{56}[i,i] = 0\,,\\
\underset{01}{Q}_{12}[i,i] = -4\frac{A}{a_i}\,, \qquad \underset{01}{Q}_{13}[i,i] = 2\frac{A}{a_i}\,, \qquad \underset{01}{Q}_{23}[i,i] = -6\frac{A}{a_i}\,,\\
\underset{01}{Q}_{32}[i,i] = -2\frac{A}{a_i}\,, \qquad \underset{01}{Q}_{33}[i,i] = -3\frac{A}{a_i}\,, \qquad \underset{01}{Q}_{34}[i,i] = 5\frac{A}{a_i}\,.
\end{gathered} \tag{7.27}$$

One may easily find from (7.5) the diagonal matrices $\underset{01}{S}_{\kappa\nu}$ with the elements

$$\begin{gathered}
\underset{01}{S}_{11}[i,i] = \underset{01}{S}_{12}[i,i] = \underset{01}{S}_{21}[i,i] = \underset{01}{S}_{22}[i,i] = \underset{01}{S}_{31}[i,i] = 0\,,\\
\underset{01}{S}_{13}[i,i] = -4\frac{A}{a_i}\,, \qquad \underset{01}{S}_{23}[i,i] = -6\frac{A}{a_i}\,, \qquad \underset{01}{S}_{32}[i,i] = 2\frac{A}{a_i}\,,\\
\underset{01}{S}_{34}[i,i] = -\frac{5}{2}\frac{A}{a_i}\,, \qquad \underset{01}{S}_{55}[i,i] = \underset{01}{S}_{56}[i,i] = 0\,.
\end{gathered} \tag{7.28}$$

There results

$$\underset{01}{G}^*_{12}[i,i] = -4\frac{A}{a_i}\,, \qquad \underset{01}{G}^*_{33}[i,i] = -3\frac{A}{a_i}\,. \tag{7.29}$$

Therefore, in the expansions (7.25) matrix $\underset{01}{A}$ is equal to $\underset{01}{G}^*_{33}$ and $\underset{01}{B} = 0$. Since

$$\alpha_k = e_k \exp(-\mathrm{i}\,\pi_k) + \ldots\,, \qquad \beta_k = \sin i_k \exp(-\mathrm{i}\,\Omega_k) + \ldots \tag{7.30}$$

(the relationships between the elements α_k, β_k and Kepler elements e_k, i_k, π_k, Ω_k are given in more detail in Brumberg, 1995) it is easy to see that the frequencies

$$\nu_i = 3\sigma\frac{A}{a_i}n_i \tag{7.31}$$

determine the Schwarzschild advances of the planetary perihelia (neglecting the squares of the eccentricities).

It is interesting to compare the solution (7.28) with the traditional solution of the Schwarzschild problem. The orbital coordinates in the Schwarzschild problem

$$x + \mathrm{i}\,y = r \exp \mathrm{i}\,u \tag{7.32}$$

within the terms of the first power in the eccentricities are determined by the expressions (Brumberg, 1991)

$$\frac{r}{a} = 1 - \frac{m}{a} - \left(1 - 2\frac{m}{a}\right) e\cos(\lambda - \pi) + \dots\,, \tag{7.33}$$

$$u = \lambda + 2e\sin(\lambda - \pi) + \dots\,. \tag{7.34}$$

Here

$$\dot\lambda = n\,, \qquad \dot\pi = 3\frac{m}{a}n\,, \tag{7.35}$$

m being the gravitational radius of the central body, n and a being related by the third Kepler law. Rewriting (7.32) in the form (3.2), one gets

$$p = \frac{m}{a} + e\left(-\frac{1}{2} + 0\cdot\frac{m}{a}\right)\exp \mathrm{i}(\lambda - \pi) + e\left(-\frac{3}{2} - 2\frac{m}{a}\right)\exp\left[-\mathrm{i}(\lambda-\pi)\right] + \dots\,. \tag{7.36}$$

On the other hand, with the aid of (7.11), (7.12) and (7.28) one gets

$$\underset{01}{c}[i,i] = -\frac{15}{4}\frac{A}{a_i} + \frac{1}{2}\underset{01}{S}_{33}[i,i]\,, \qquad \underset{01}{d}[i,i] = \frac{37}{4}\frac{A}{a_i} - \frac{3}{2}\underset{01}{\bar S}_{33}[i,i]\,. \tag{7.37}$$

Considering the structure of the equations (7.5) it is seen that the matrices S_{33} and S_{55} are determined up to arbitrary constant diagonal matrices serving as the constants of integration. The simplest option seems to annul these constants. But sometimes another option might be reasonable. For example, to retain the same meaning of the integration constant as in (7.36) one should put

$$\underset{01}{S}_{33}[i,i] = \frac{15}{2}\frac{A}{a_i}\,, \tag{7.38}$$

resulting to

$$\underset{01}{c}[i,i] = 0\,, \qquad \underset{01}{d}[i,i] = -2\frac{A}{a_i}\,. \tag{7.39}$$

The expressions (6.5) and (7.38) are in complete correspondence with (7.36).

The matrices $\underset{11}{K}$, $\underset{11}{L}$, $\underset{11}{K'}$, $\underset{11}{L'}$, $\underset{11}{M}$ and $\underset{11}{M'}$ of the terms of the order $\mu\sigma$ are found by comparing (4.8) and (4.9) with (5.1) and (5.2). The explicit expressions for these matrices are given in the Appendix. These matrices are of particular interest for investigating of the secular system. Based on (7.3) the general formula

$$G_{\kappa\nu}[i,j] = Q_{\kappa\nu}[i,j] + \sum_{\lambda=1}^{6}\sum_{k=1}^{N}\Big(Q_{\kappa\lambda}[i,k]S_{\lambda\nu}[k,j] - \frac{n_k}{n_i}S_{\kappa\lambda}[i,k]G^*_{\lambda\nu}[k,j]\Big) \tag{7.40}$$

involves

$$\underset{11}{G}_{\kappa\nu}[i,j] = \underset{11}{Q}_{\kappa\nu}[i,j] + \sum_{\lambda=1}^{6}\sum_{k=1}^{N}\Big(\underset{10}{Q}_{\kappa\lambda}[i,k]\,\underset{01}{S}_{\lambda\nu}[k,j] + \underset{01}{Q}_{\kappa\lambda}[i,k]\,\underset{10}{S}_{\lambda\nu}[k,j] -$$

$$- \frac{n_k}{n_i}\underset{10}{S}_{\kappa\lambda}[i,k]\,\underset{01}{G^*_{\lambda\nu}}[k,j] - \frac{n_k}{n_i}\underset{01}{S}_{\kappa\lambda}[i,k]\,\underset{10}{G^*_{\lambda\nu}}[k,j]\Big)\,. \tag{7.41}$$

By substituting the terms of the order μ from the Newtonian theory and the terms of the order σ and $\mu\sigma$ by presented here algorithms one gets the matrix $\underset{11}{G}_{33}$. More specifically,

$$\underset{11}{G}_{33}[i,j] = \underset{11}{Q}_{33}[i,j]\,, \qquad i \neq j \tag{7.42}$$

and

$$\begin{aligned} \underset{11}{G}_{33}[i,i] = \underset{11}{Q}_{33}[i,i] + \frac{A}{a_i}\Big(&-6\,\underset{10}{Q}_{32}[i,i] + \frac{15}{2}\,\underset{10}{Q}_{33}[i,i] - \frac{5}{2}\,\underset{10}{Q}_{34}[i,i] - \\ &- 2\,\underset{10}{S}_{23}[i,i] + 5\,\underset{10}{S}_{43}[i,i] - \frac{15}{2}\,\underset{10}{G}^{*}_{33}[i,i]\Big)\,. \end{aligned} \tag{7.43}$$

For determining $\underset{11}{G}_{55}$ it is sufficient to take into account only the first term in the right–hand member (7.41) coinciding with Q_{55} from (4.30). In contrast to the Newtonian theory these matrices are not real.

8. **Conclusion**

This paper presents two techniques to determine the relativistic planetary perturbations in the theories of motion of the major planets. The first technique is of practical orientation enabling one to find by iterations the relativistic terms caused by perturbing planetary actions (direct and indirect) in the form used in the classical theories of motion. In particular, this technique may permit to complement the VSOP theories of the major planets by taking into account the relativistic direct planetary perturbations. The second technique extends the general planetary theory for the relativistic problem of planetary motions. This technique involves the separation of the terms dependent on fast changing variables (the mean longitudes of the planets) and the terms describing the secular evolution of the planetary orbits. The algorithms are developed to compute the main relativistic planetary perturbations not dependent on the eccentricities and inclinations as well as the relativistic terms in the matrices of the secular system. The actual computations and further theoretical investigation of the secular system remain to be performed in the future.

Originally this paper was published in Russian (Brumberg, 1999) without Appendix.

References

Bretagnon P. and Francou G., 1988. Planetary Theories in Rectangular and Spherical Variables. VSOP87 Solutions. Astron. Astrophys., **202**, 309–315.

Brumberg V. A., 1995. Analytical Techniques of Celestial Mechanics. Springer, Heidelberg.

Brumberg V. A. 1991. Essential Relativistic Celestial Mechanics. Hilger, Bristol.

Brumberg V.A. 1999. Some Techniques for Determining Relativistic Planetary Perturbations in the Theories of Motion of the Major Planets. Inst. Appl. Astron. Transactions, **4**, 199–224, St. Petersburg (in Russian)

Appendix

This Appendix contains in (A.1)–(A.17) the coefficients of the right–hand members of section 5 and in (A.18)–(A.29) the matrices of section 7 needed to construct the secular system taking into account the terms of the order $\mu\sigma$.

$$A_{i001} = \frac{n_i^2}{a_i}\kappa_{ii} + n_i^2 \sum_{j\neq i} \kappa_{jj}\left[\frac{3}{2a_j} + \frac{7}{4}\frac{a_i^2}{a_j^3} + \frac{1}{8}\frac{a_i}{a_j^2}\left(1 - 4\frac{n_i}{n_j}\right)\zeta_{ij} + \right.$$

$$\left. + \frac{3}{8}\frac{a_i}{a_j^2}\left(1 - 4\frac{n_i}{n_j}\right)\zeta_{ij}^{-1} + \frac{6}{\Delta_{ij}} + \left(2 - \frac{7}{4}\frac{a_i^3}{a_j^3}\right)\frac{a_i(a_i - a_j\zeta_{ij})}{\Delta_{ij}^3}\right], \qquad (A.1)$$

$$A_{i003} = 2\frac{n_i^2}{a_i}\kappa_{ii} + \frac{n_i}{a_i^2}\sum_{j\neq i}\kappa_{jj} n_j a_j \zeta_{ij}\,, \qquad (A.2)$$

$$A_{i0j1} = n_i^2\kappa_{jj}\left[\frac{1}{2a_j} + \frac{21}{4}\frac{a_i^2}{a_j^3} + \frac{3}{8}\frac{a_i}{a_j^2}\zeta_{ij} + \frac{1}{8}\frac{a_i}{a_j^2}\left(1 + 8\frac{n_i}{n_j}\right)\zeta_{ij}^{-1} - \frac{21}{4}\frac{a_i^3}{a_j^3}\frac{1}{\Delta_{ij}} + \right.$$

$$\left. + \left(2 - \frac{7}{4}\frac{a_i^3}{a_j^3}\right)\frac{a_j(a_j - a_i\zeta_{ij}^{-1})}{\Delta_{ij}^3}\right], \qquad (A.3)$$

$$A_{i0j3} = \frac{n_i}{a_i^2}\kappa_{jj} n_j a_j \zeta_{ij}^{-1}\,, \qquad (A.4)$$

$$A_{ij01} = -\frac{7}{4}\frac{n_j^2}{a_i} + \left[\left(3n_i^2 + \frac{5}{2}n_j^2\right)a_i - \left(2n_i + \frac{9}{4}n_j\right)n_j a_j \zeta_{ij}\right]\frac{a_i}{\Delta_{ij}^3} +$$

$$+ \frac{3}{4}n_j^2\frac{a_i^2 a_j^2}{\Delta_{ij}^5}\left(\zeta_{ij}^2 - 1\right) + \frac{3}{2}\frac{a_i(a_i - a_j\zeta_{ij})}{\Delta_{ij}^5}\left[\left(3n_i^2 - \frac{1}{2}n_j^2\right)a_i^2 - n_j^2 a_j^2 + \right.$$

$$\left. + \left(2n_i + \frac{1}{4}n_j\right)n_j a_i a_j\left(\zeta_{ij} + \zeta_{ij}^{-1}\right) - \frac{5}{8}n_j^2\frac{a_i^2 a_j^2}{\Delta_{ij}^2}\left(\zeta_{ij}^2 + \zeta_{ij}^{-2} - 2\right)\right], \qquad (A.5)$$

$$A_{ij03} = n_i^2\frac{a_i^2}{\Delta_{ij}^3} - 2n_i n_j\frac{a_i a_j}{\Delta_{ij}^3}\zeta_{ij}\,, \qquad (A.6)$$

$$A_{ijj1} = -\frac{21}{4}\frac{n_j^2}{a_i} + 6\frac{n_j^2}{\Delta_{ij}} + \left[\frac{9}{2}n_j^2 a_j^2 - \frac{3}{4}n_j^2 a_i^2 + \frac{3}{8}n_j^2 a_i a_j \zeta_{ij} - \right.$$

$$\left. - \left(2n_i + \frac{15}{8}\right)n_j a_i a_j \zeta_{ij}^{-1}\right]\frac{1}{\Delta_{ij}^3} + \frac{3}{4}n_j^2\frac{a_i^2 a_j^2}{\Delta_{ij}^5}\left(\zeta_{ij}^{-2} - 1\right) +$$

$$+ \frac{3}{2}\frac{a_j(a_j - a_i\zeta_{ij}^{-1})}{\Delta_{ij}^5}\left[\left(2n_i + \frac{1}{4}n_j\right)n_j a_i a_j\left(\zeta_{ij} + \zeta_{ij}^{-1}\right) + \right.$$

$$\left. + 3n_i^2 a_i^2 - \frac{1}{2}n_j^2 a_i^2 - n_j^2 a_j^2 - \frac{5}{8}n_j^2\frac{a_i^2 a_j^2}{\Delta_{ij}^2}\left(\zeta_{ij}^2 + \zeta_{ij}^{-2} - 2\right)\right], \qquad (A.7)$$

$$A_{ijj3} = 2n_j^2 \frac{a_j^2}{\Delta_{ij}^3} - 2n_i n_j \frac{a_i a_j}{\Delta_{ij}^3}\zeta_{ij}^{-1} + \frac{3}{4} n_j^2 a_i a_j^2 \frac{a_i - a_j\zeta_{ij}}{\Delta_{ij}^5}\Big(\zeta_{ij}^{-2} - 1\Big), \tag{A.8}$$

$$B_{i001} = \frac{\mathrm{i}}{4a_i^2} \sum_{j\neq i} \kappa_{jj} n_j a_j (\zeta_{ij} - 3\zeta_{ij}^{-1}), \tag{A.9}$$

$$B_{i003} = \frac{\mathrm{i}}{2a_i}\kappa_{ii} n_i \, , \tag{A.10}$$

$$B_{i0j1} = \frac{\mathrm{i}}{2a_i^2}\kappa_{jj} n_j a_j \zeta_{ij}^{-1} \, , \tag{A.11}$$

$$B_{i0j3} = \frac{\mathrm{i}}{2a_i^2}\kappa_{jj} n_j a_j \zeta_{ij}^{-1} \, , \tag{A.12}$$

$$B_{ij01} = \mathrm{i}\, \frac{a_i a_j}{\Delta_{ij}^3}\Big(\frac{3}{2}n_j - 2n_i\Big)\Big[-\zeta_{ij} + \frac{3}{2}\frac{a_i(a_i - a_j\zeta_{ij})}{\Delta_{ij}^2}(\zeta_{ij} - \zeta_{ij}^{-1})\Big], \tag{A.13}$$

$$B_{ij03} = -2\,\mathrm{i}\, n_i \frac{a_i(a_i - a_j\zeta_{ij})}{\Delta_{ij}^3} \, , \tag{A.14}$$

$$B_{ijj1} = \mathrm{i}\, \frac{a_i a_j}{\Delta_{ij}^3}\Big(\frac{3}{2}n_j - 2n_i\Big)\Big[\zeta_{ij}^{-1} + \frac{3}{2}\frac{a_j(a_j - a_i\zeta_{ij}^{-1})}{\Delta_{ij}^2}(\zeta_{ij} - \zeta_{ij}^{-1})\Big], \tag{A.15}$$

$$B_{ijj3} = -\frac{3}{2}\,\mathrm{i}\, n_j \frac{a_j(a_j - a_i\zeta_{ij}^{-1})}{\Delta_{ij}^3} \, . \tag{A.16}$$

One may add therewith the relations of conjugation

$$\begin{aligned}
&A_{i002} = \bar{A}_{i001}\, , \; A_{i004} = -\bar{A}_{i003}\, , \; A_{i0j2} = \bar{A}_{i0j1}\, , \; A_{i0j4} = -\bar{A}_{i0j3}\, , \\
&A_{ij02} = \bar{A}_{ij01}\, , \; A_{ij04} = -\bar{A}_{ij03}\, , \; A_{ijj2} = \bar{A}_{ijj1}\, , \; A_{ijj4} = -\bar{A}_{ijj3}\, , \\
&B_{i002} = \bar{B}_{i001}\, , \; B_{i004} = -\bar{B}_{i003}\, , \; B_{i0j2} = \bar{B}_{i0j1}\, , \; B_{i0j4} = -\bar{B}_{i0j3}\, , \\
&B_{ij02} = \bar{B}_{ij01}\, , \; B_{ij04} = -\bar{B}_{ij03}\, , \; B_{ijj2} = \bar{B}_{ijj1}\, , \; B_{ijj4} = -\bar{B}_{ijj3}\, .
\end{aligned} \tag{A.17}$$

$$\begin{aligned}
\underset{11}{K}_{ii} = &-\frac{3}{4}\Big(\underset{11}{p}_i^{(0)} + \underset{11}{q}_i^{(0)}\Big) + \frac{A}{a_i}\Big[\frac{21}{4}\Big(\underset{10}{p}_i^{(0)} + \underset{10}{q}_i^{(0)}\Big) + \frac{1}{2\,\mathrm{i}\, n_i}\Big(3\underset{10}{\dot{p}}_i^{(0)} - \underset{10}{\dot{q}}_i^{(0)}\Big) - \\
&- \sum_{j\neq i}\kappa_{jj}\Big(2\psi_{200000}^{(ij)} + \psi_{110000}^{(ij)} + \frac{a_i}{a_j}\psi_{101000}^{(ij)} + \frac{a_i}{a_j}\psi_{100100}^{(ij)}\Big)\Big] + \\
&+ \frac{A}{n_i^2}\Big\{-A_{i0}^{*\,(0)} - \kappa_{ii}A_{0i}^{(0)} - \mathrm{i}\, n_i(B_{i0}^{*\,(0)} + \kappa_{ii}B_{0i}^{(0)}) + \\
&+ A_{i001} + \mathrm{i}\, n_i B_{i001} + 9\kappa_{ii}\frac{n_i^2}{a_i} + \sum_{j\neq i}\kappa_{jj}\Big[-A_{ij}^{(0)} - \mathrm{i}\, n_i B_{ij}^{(0)} + \\
&+ \Big(1 - \frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ij01} + \mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ij01}\Big]\Big\},
\end{aligned} \tag{A.18}$$

$$\underset{11}{K}_{ij} = -\frac{A}{a_i}\kappa_{jj}\Big(\psi^{(ij)}_{101000} + \psi^{(ij)}_{011000} + 2\frac{a_i}{a_j}\psi^{(ij)}_{002000} + \frac{a_i}{a_j}\psi^{(ij)}_{001100}\Big) + \frac{A}{n_i^2}\Big\{A_{i0j1} +$$

$$+\,\mathrm{i}\,n_i B_{i0j1} + \kappa_{jj}\Big[-\Big(A^{(0)}_{0j} - A^{(0)}_{ij}\Big)\frac{a_j}{a_i}\zeta_{ij}^{-1} - \mathrm{i}\,n_j\Big(B^{(0)}_{0j} - B^{(0)}_{ij}\Big)\frac{a_j}{a_i}\zeta_{ij}^{-1} +$$

$$+\,9\frac{n_j^2}{a_i}\zeta_{ij}^{-1} + \Big(1 - \frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ijj1} + \mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ijj1}\Big]\Big\}, \qquad (A.19)$$

$$\underset{11}{L}_{ii} = -\frac{3}{4}\Big(\underset{11}{p}_i^{(0)} + 5\,\underset{11}{q}_i^{(0)}\Big) + \frac{A}{a_i}\Big[\frac{3}{4}\Big(7\,\underset{10}{p}_i^{(0)} + 11\,\underset{10}{q}_i^{(0)}\Big) + \frac{3}{2\,\mathrm{i}\,n_i}\Big(\underset{10}{\dot{p}}_i^{(0)} + \underset{10}{\dot{q}}_i^{(0)}\Big) -$$

$$-\sum_{j\neq i}\kappa_{jj}\Big(\psi^{(ij)}_{110000} + 2\psi^{(ij)}_{020000} + \frac{a_i}{a_j}\psi^{(ij)}_{011000} + \frac{a_i}{a_j}\psi^{(ij)}_{010100}\Big)\Big] +$$

$$+\,\frac{A}{n_i^2}\Big\{A_{i002} + \mathrm{i}\,n_i B_{i002} + 9\kappa_{ii}\frac{n_i^2}{a_i} + \sum_{j\neq i}\Big[\Big(1 - \frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ij02} +$$

$$+\,\mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ij02}\Big]\Big\}, \qquad (A.20)$$

$$\underset{11}{L}_{ij} = -\frac{A}{a_i}\kappa_{jj}\Big(\psi^{(ij)}_{100100} + \psi^{(ij)}_{010100} + \frac{a_i}{a_j}\psi^{(ij)}_{001100} + 2\frac{a_i}{a_j}\psi^{(ij)}_{000200}\Big) +$$

$$+\,\frac{A}{n_i^2}\Big\{A_{i0j2} + \mathrm{i}\,n_i B_{i0j2} + \kappa_{jj}\Big[9\frac{n_j^2}{a_i}\zeta_{ij}^{-1} + \Big(1 - \frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ijj2} +$$

$$+\,\mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ijj2}\Big]\Big\}, \qquad (A.21)$$

$$\underset{11}{K'}_{ii} = \frac{A}{a_i}\Big(3\,\underset{10}{p}_i^{(0)} + \frac{4}{\mathrm{i}\,n_i}\underset{10}{\dot{p}}_i^{(0)} + \frac{2}{\mathrm{i}\,n_i}\underset{10}{\dot{q}}_i^{(0)}\Big) - \frac{A}{n_i^2}\Big\{-\mathrm{i}\,n_i\Big(B^{*(0)}_{i0} + \kappa_{ii}B^{(0)}_{0i}\Big) +$$

$$+\,\frac{7}{2}\kappa_{ii}\frac{n_i^2}{a_i} + A_{i003} + \mathrm{i}\,n_i B_{i003} + \sum_{j\neq i}\kappa_{jj}\Big[-\mathrm{i}\,n_i B^{(0)}_{ij} +$$

$$+\,\Big(1 - \frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ij03} + \mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ij03}\Big]\Big\}, \qquad (A.22)$$

$$\underset{11}{K'}_{ij} = -\frac{A}{n_i^2}\Big\{A_{i0j3} + \mathrm{i}\,n_i B_{i0j3} + \kappa_{jj}\Big[-\mathrm{i}\,n_j\big(B^{(0)}_{0j} - B^{(0)}_{ij}\big)\frac{a_j}{a_i}\zeta_{ij}^{-1} +$$

$$+\,\frac{7}{2}\frac{n_j^2}{a_i}\zeta_{ij}^{-1} + \Big(1 - \frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ijj3} + \mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ijj3}\Big]\Big\}, \qquad (A.23)$$

$$\underset{11}{L'}_{ii} = -\frac{A}{a_i}\Big(\underset{10}{p}{}_i^{(0)} + \frac{2}{\mathrm{i}\,n_i}\underset{10}{\dot{p}}{}_i^{(0)}\Big) + \frac{A}{n_i^2}\Big\{-\frac{1}{2}\kappa_{ii}\frac{n_i^2}{a_i} + A_{i004} + \mathrm{i}\,n_i B_{i004} +$$

$$+\sum_{j\neq i}\kappa_{jj}\Big[\Big(1-\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ij04} + \mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ij04}\Big]\Big\}, \qquad (A.24)$$

$$\underset{11}{L'}_{ij} = \frac{A}{n_i^2}\Big\{A_{i0j4} + \mathrm{i}\,n_i B_{i0j4} + \kappa_{jj}\Big[-\frac{1}{2}\frac{n_j^2}{a_i}\frac{a_j}{a_i}\zeta_{ij}^{-1} + \Big(1-\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)A_{ijj4} +$$

$$+\,\mathrm{i}\Big(n_i - n_j\frac{a_j}{a_i}\zeta_{ij}^{-1}\Big)B_{ijj4}\Big]\Big\}, \qquad (A.25)$$

$$\underset{11}{M}_{ii} = \frac{3}{2}\Big(\underset{11}{p}{}_i^{(0)} + \underset{11}{q}{}_i^{(0)}\Big) - \frac{A}{a_i}\Big[\frac{3}{2}\Big(\underset{10}{p}{}_i^{(0)} + \underset{10}{q}{}_i^{(0)}\Big) + \frac{1}{\mathrm{i}\,n_i}\Big(\underset{10}{\dot{p}}{}_i^{(0)} - \underset{10}{\dot{q}}{}_i^{(0)}\Big) +$$

$$+\sum_{j\neq i}\kappa_{jj}\Big(\theta^{(ij)}_{10000-1} + \theta^{(ij)}_{01000-1} + \frac{a_i}{a_j}\theta^{(ij)}_{00100-1} + \frac{a_i}{a_j}\theta^{(ij)}_{00010-1}\Big)\Big] -$$

$$-\frac{A}{n_i^2}\Big(A^{*(0)}_{i0} + \kappa_{ii}A^{(0)}_{0i} + \sum_{j\neq i}\kappa_{jj}A^{(0)}_{ij}\Big), \qquad (A.26)$$

$$\underset{11}{M}_{ij} = -\frac{A}{a_i}\kappa_{jj}\Big(\theta^{(ij)}_{100000} + \theta^{(ij)}_{010000} + \frac{a_i}{a_j}\theta^{(ij)}_{001000} + \frac{a_i}{a_j}\theta^{(ij)}_{000100}\Big) -$$

$$-\frac{A}{n_i^2}\kappa_{jj}\frac{a_j}{a_i}\Big(A^{(0)}_{0j} - A^{(0)}_{ij}\Big), \qquad (A.27)$$

$$\underset{11}{M'}_{ii} = \frac{A}{a_i}\frac{2}{\mathrm{i}\,n_i}\Big(\underset{10}{\dot{p}}{}_i^{(0)} + \underset{10}{\dot{q}}{}_i^{(0)}\Big) - \frac{A}{\mathrm{i}\,n_i}\Big(B^{*(0)}_{i0} + \kappa_{ii}B^{(0)}_{0i} + \sum_{j\neq i}\kappa_{jj}B^{(0)}_{ij}\Big), \qquad (A.28)$$

$$\underset{11}{M'}_{ij} = \mathrm{i}\,n_j\frac{A}{n_i^2}\kappa_{jj}\frac{a_j}{a_i}\Big(B^{(0)}_{0j} - B^{(0)}_{ij}\Big)\,. \qquad (A.29)$$

The coefficients $\theta_\sigma^{(ij;klrsmt)}$ occurring in $(A.26)$ and $(A.27)$ and associated with the expansion of the Newtonian right–hand members W_i of (4.5) may be easily computed (like $\psi_\sigma^{(ij;klrsmt)}$) with the aid of the hypergeometric series. The general algorithm for their computation is given by (10.2.6), (6.2.60), (6.2.51) and (2.3.4) of (Brumberg, 1995).