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Solutions de viscosité des équations de Hamilton-Jacobi et minmax itérés

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Résumé

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Dans cette thèse, nous étudions les solutions des équations Hamilton-Jacobi. Plus précisément, nous comparons la solution de viscosité, obtenue comme limite de solutions de l'équation perturbée par un petit terme de diffusion, et la solution minmax, définie géométriquement à partir d'une fonction génératrice quadratique à l'infini. Dans la littérature, il y a des cas bien connus où les deux coïncident, par exemple lorsque le hamiltonien est convexe ou concave, le minmax pouvant alors être réduit à un min ou un max. Mais les solutions minmax et de viscosité diffèrent en général. Nous construisons des "minmax itérés" en répétant pas à pas la procédure de minmax et démontrons que, quand la taille du pas tend vers zéro, les minmax itérés tendent vers la solution de viscosité. Dans une deuxième partie, nous étudions les lois de conservation en dimension un d'espace par le méthode de "front tracking". Nous montrons que dans le cas où la donnée initiale est convexe, la solution de viscosité et le minmax sont égaux. Et comme application, nous décrivons sur des exemples la manière dont sont construites les singularités de la solution de viscosité. Pour finir, nous montrons que la notion de minmax n'est pas aussi évidente qu'il y paraît.

Mots-clefs

Équation de Hamilton-Jacobi, solution de viscosité, famille génératrice, minmax, minmax itéré, front d'onde

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Viscosity solutions of Hamilton-Jacobi equation and iterated minmax

Abstract

In this thesis, we study the solutions of Hamilton-Jacobi equations. We will compare the viscosity solution and the minmax solution, with the latter defined by a geometric method. In the literature, there are well-known cases where these two solutions coincide: if the Hamiltonian is convex or concave with respect to the momentum variable, the

minmax can be reduced to min or max. The minmax and viscosity solutions are different in general. We will construct "iterated minmax" by iterating the minmax step by step and prove that, as the size of steps go to zero, the iterated minmax converge to the viscosity solution. In particular, we study the equations of conservation laws in dimension one, where, by the "front tracking" method, we shall see that in the case where the initial function is convex, the viscosity solution and the minmax are equal. And as an application, we use the limiting iterated process to describe the singularities of the viscosity solution. In the end, we show that the notion of minmax is not so obvious.

Keywords

Hamilton-Jacobi equation, viscosity solution, generating familly, minmax, iterated minmax, wave front

Contents

Introduction

Cette thèse concerne l'étude du problème de Cauchy

$$\text{(H-J)} \qquad \begin{cases} \partial_t u(t,x) + H(t,x,\partial_x u) = 0, \\ u(0,x) = v(x) \end{cases}$$

pour l'équation de Hamilton-Jacobi. La solution de viscosité dans la théorie analytique des équations aux dérivées partielles sera approchée par une méthode géométrique.

Solution de viscosité

Même si v est C^∞, le problème (H-J) n'a pas en général de solution globale C^1. Cela conduit à chercher des solutions faibles, par exemple les fonctions qui vérifient l'équation presque partout. Cependant, cette notion ne suffit pas à assurer l'unicité. L'exigence d'unicité d'une solution ayant (éventuellement) un sens physique demande d'imposer une condition supplémentaire sur les singularités des solution faibles. Parmi les efforts faits dans ce sens, la notion générale de solutions de viscosité introduite en 1981 par M.G. Crandall et P.L. Lions a montré sa valeur pour établir l'existence, l'unicité et la stabilité au sens le plus général. De nombreux travaux ont contribué à faire mûrir cette théorie du côté analytique.

Solution géométrique et variationelle

Le problème (H-J) a une unique solution "multiforme" globale, définie par la méthode des caractéristiques: on considère l'équation comme une hypersurface dans le cotangent $T^*(\mathbb{R} \times M)$. La réunion des caractéristiques de cette hypersurface issues de la sous-variété isotrope initiale définie par la dérivée de v est une sous-variété lagrangienne contenue dans cette hypersuface : c'est la solution géométrique de (H-J), c'est-à-dire le "graphe de la dérivée" de la solution multiforme. Lorsque (par exemple) le hamiltonien H est à support compact, cette sous-variété lagrangienne est l'image de la section nulle du cotangent par temps 1 d'une isotopie hamiltonienne et elle admet une famille génératrice quadratique à l'infini (FGQI).

On peut sélectionner une section de la projection d'une telle sous-variété L sur la base $\mathbb{R} \times M$ ("graph selector") en prenant le minmax d'une FGQI par rapport aux variables supplémentaires; il résulte d'un théorème de Viterbo et Théret que cette section ne dépend que de L et non de la FGQI choisie (ni donc de le problème (H-J) dont L est solution géométrique). Elle a été proposée comme une construction géométrique de solution faible pour les équation (H-J) non convexes par M. Chaperon [21], suivi de T. Joukovskaïa, A. Ottolenghi, C. Viterbo, F. Cardin, [45, 63, 64, 65, 12]. Récemment, elle apparaît dans les travaux de M.-C. Arnaud, P. Bernard, J. Santos [2, 11, 10] sur la théorie de KAM faible. D'autre part, par sa nature géométrique, elle devient un outil de la topologie symplectique, développé par C.Viterbo [62]. Elle apparaît aussi comme un lien entre la topologie symplectique et la théorie d'Aubry-Mather chez G. Paternain, L. Polterovich, et K. Siburg [53].

Le contenu de cette thèse est organisé comme suit:

Dans le chapitre 1, nous présentons la théorie générale des FGQI. Des formules explicites sont donnés en utilisant les fonctions génératrices obtenues par la méthode de "géodésiques brisées" discrétisant la fonctionnelle d'action du calcul variationnel pour la ramener de la dimension infinie à la dimension finie. L'existence de FGQI est discutée dans le cas où la variété n'est pas compacte. Le minmax est introduit en language homologique. Puis nous étendons ces objets de classe C^2 au cas Lipschitzien, en vue de l'application aux minmax itérés. Nous obtenons ainsi un sélecteur généralisé donné par le minmax.

Dans le chapitre 2, nous comparons les solutions de viscosité et minmax de l'équation de Hamilton-Jacobi. Nous traitons à nouveau les cas convexes classiques par le minmax, qui est en fait ici réduit au min. Ce sont des cas où la solution minmax et la solution de viscosité coïncident. Le fait qu'elles soient solutions de viscosité vient de ce qu'elles possèdent la "propriété de semi-groupe" par rapport au temps: nous démontrons en effet que si les solutions minmax possèdent cette propriété, ce sont les solutions de viscosité (Proposition 2.44).

Puis nous introduisons le minmax itéré dans le cas où le minmax ne définit pas un "semi-groupe". Nous démontrons que la limite du minmax itéré est la solution de viscosité. Il s'avère que le sélecteur minmax se comporte comme un "générateur" définie par P.E. Souganidis dans [58], et notre procédure s'inscrit dans ses schèmes d'approximation générales aux solutions de viscosités . La vertu de l'approximation par minmax itérées , c'est qu'en raison de sa propriété géométrique, elle pourrait nous fournir une description géométrique de la solution de viscosité. Après, en particulier, nous étudions l'équation des lois de conservation de dimension un, où, par le méthode de "front tracking", Nous montrons que dans le cas où la donnée initiale est convexe, la solution de viscosité et le minmax sont égaux.

Dans la dernière partie nous utilisons notre résultat pour décrire sur des exemples la formation des singularités des solutions de viscosité des lois de conservation. Quand il y a raréfaction, la passage à la limite dans les minmax itérés nous explique d'où vient la partie de la solution de viscosité qui n'est pas contenue dans la solution géométrique: elle vient de la partie "verticale" de la différentielle de Clarke ∂u, qui décrit la singularité de la différentielle du à chaque pas des minmax itérés. C'est aussi une explication du fait que le minmax ne possède pas la propriété de semi-groupe. Ce phénomène ne peut pas se produire dans les cas convexes, où la dérivée ordinaire du suffit à rendre compte de tout.

Dans le chapitre 3, nous nous intéressons au sélecteur minmax lui-même, en particulier aux questions suivantes:

1. Est-ce que le minmax et son analogue maxmin sont égaux?
2. Est-ce qu'il y a un seul "minmax"? Plus précisement, est-ce que le minmax dépend du choix des coefficients de l'homologie qui le définit?

Nous montrons que minmax et maxmin sont égaux s'ils sont définis par l'homologie à coefficients dans un corps. Cependant, un contre-exemple donné par F. Laudenbach nous dit que ceci cesse d'être vrai si les coefficients appartiennent à un anneau quelconque et que, dans le cas d'un corps, le minmax-maxmin dépend du choix du corps.

Chapter 1

Generating families and minmax selector

We will first give a brief survey of the classical theory, for a closed manifold M, of generating families for Lagrangian submanifolds $L \subset T^*M$ and hereafter the minmax selector which serve to extract a section from the Lagrangian. Then we pass to the model case $M = \mathbb{R}^d$ where we will generalize the classical notions, on the one hand, to fit the non-compactness of manifolds, and on the other hand, to the Lipschitz case where we do not have smooth Lagrangian submanifolds, but the notion of generating family and minmax still hold for a similar objet.

1.1 General theory for closed manifolds

Definition 1.1. A *generating family* for a Lagrangian submanifold $L \subset T^*M$ is a C^2 function $S : M \times \mathbb{R}^k \to \mathbb{R}$ such that 0 is a regular value of the map $(x, \eta) \mapsto \partial S(x, \eta)/\partial \eta$ and

$$L = \left\{ (x, \frac{\partial S}{\partial x}(x, \eta)) : \frac{\partial S}{\partial \eta} = 0 \right\};$$

more precisely, the condition that 0 is a regular value implies that the *critical locus* $\Sigma_S := \{(x, \eta) | \partial_\eta S = 0\}$ is a submanifold and that the map

$$i_S : \Sigma_S \to T^*M, \quad (x, \eta) \mapsto (x, \partial_x S(x, \eta))$$

is an immersion; we require that i_S be an embedding and, of course, $i_S(\Sigma_S) = L$.

A function S on $M \times \mathbb{R}^k$ need not have critical points. However, it does have critical points if we prescribe some behavior at infinity as in the following definition:

Definition 1.2. A generating family $S : M \times \mathbb{R}^k \to \mathbb{R}$ is *(exactly) quadratic at infinity* if

$$S(x, \eta) = \psi(x, \eta) + Q(\eta)$$

where Q is a nondegenerate quadratic form and $S = Q$ outside a compact set.

The existence of a GFQI is invariant under Hamiltonian isotopy [1]:

1. Recall that an *isotopy* of T^*M is a smooth path $(g_t)_{t\in[0,1]}$ in the group of diffeomorphisms of T^*M onto itself. Such an isotopy is called *symplectic* when each g_t preserves the canonical symplectic form ω_M; by the Cartans' formula $\mathcal{L}_X \omega_M = (d\omega_M)X + d(\omega_M X)$, since $d\omega_M = 0$, this amounts to saying that the infinitesimal generator $X_t = (\frac{d}{dt} g_t) \circ g_t^{-1}$ of the isotopy is such that the interior product $\omega_M X_t$ is a closed 1-form for all t. When this 1-form is exact, the vector fields X_t and the isotopy are called *Hamiltonian*.

Theorem 1.3 (Sikorav [57])**.** *Suppose L_0 and L_1 are two Lagrangian submanifolds of T^*M which are Hamiltonianly isotopic, and L_0 admits a GFQI, then so does L_1. In particular, any Lagrangian manifold Hamiltonianly isotopic to the zero section 0_{T^*M} admits a GFQI.*

Note that the generating families are not unique. Let $S : M \times \mathbb{R}^k \to \mathbb{R}$ be a generating family of L, then one can obtain another family $\tilde{S}$ generating the same L by

(a) Fiberwise diffeomorphism : $\tilde{S}(x, \eta) := S(x, \varphi(x, \eta))$, where $(x, \eta) \mapsto (x, \varphi(x, \eta))$ is a fiberwise diffeomorphism.

(b) Adding a constant: $\tilde{S}(x, \eta) := S(x, \eta) + C$.

(c) Stabilization: $\tilde{S}(x, \eta, \xi) := S(x, \eta) + q(\xi)$, where q is a nondegenerate quadratic form.

Theorem 1.4 (Viterbo, Théret [60])**.** *If a Lagrangian submanifold $L \subset T^*M$ is Hamiltonianly isotopic to the zero section 0_{T^*M}, then L admits a unique GFQI up to the above operations.*

Now given a Lagrangian submanifold $L \subset T^*M$ with a GFQI

$$S : M \times \mathbb{R}^k \to \mathbb{R}, \quad S(x, \eta) = \psi(x, \eta) + Q(\eta)$$

consider the sub-level sets

$$S_x^a := \{\eta : S(x, \eta) \leq a\},$$

the homotopy type of (S_x^a, S_x^{-a}) does not depend on a when a is large enough, we may write it as $(S_x^\infty, S_x^{-\infty})$. If the Morse index of Q is k_∞, then

$$H_i(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2) = H_i(Q^\infty, Q^{-\infty}; \mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2, & i = k_\infty \\ 0, & \text{otherwise} \end{cases}$$

Definition 1.5. The *minmax* function is defined as

$$R_S(x) := \inf_{[\sigma]=A} \max_{\eta \in \sigma} S(x, \eta)$$

where A is a generator of the homology group $H_{k_\infty}(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2)$. A relative cycle σ of class A is called a descending cycle.

We can also introduce the maxmin function by considering the homology group defined by upper level sets:

$$H_{k'_\infty}(X \setminus S_x^{-\infty}, X \setminus S_x^\infty; \mathbb{Z}_2) \simeq \mathbb{Z}_2$$

where $k'_\infty = k - k_\infty$ and $X = \mathbb{R}^k$ is the fiber space.

Definition 1.6. The *maxmin* function is defined as

$$P_S(x) := \sup_{[\sigma]=B} \min_{\eta \in \sigma} S(x, \eta)$$

where B is a generator of the homology group $H_{k'_\infty}(X \setminus S_x^{-\infty}, X \setminus S_x^\infty; \mathbb{Z}_2)$. A relative cycle σ of class B is called an ascending cycle.

Remark 1.7. The minmax and maxmin are defined fiberwise for generating families. We remark that they are well-defined for functions $f : X \to \mathbb{R}$ "quadratic at infinity" in the sense that the critical set of f is compact and $(f^\infty, f^{-\infty})$ has the homotopy type of $(Q^\infty, Q^{-\infty})$ for a nondegenerate quadratic form Q. For example, the condition is satisfied if the derivative of $f - Q$ is bounded. This is a simple generalization of functions exactly quadratic at infinity which requires $f = Q$ outside a compact set.

Remark 1.8. By the uniqueness theorem 1.4, for a given Lagrangian submanifold L, the minmax and maxmin are independent of the GFQI, up to a constant.

Proposition 1.9. *The minmax and the maxmin are equal, i.e.* $R_S(x) = P_S(x)$.

This is a particular case of Theorem 3.11 p. 71. The coefficients of the homology and cohomology groups are taken in $\mathbb{Z}_2$, which is a field, a crucial point for the coincidence of minmax and maxmin. They may indeed differ, for general functions quadratic at infinity, when the coefficients are in $\mathbb{Z}$ (and they also depend on the fields), see Chapter 3.

Lemma 1.10. *The minmax* $R_S(x)$ *is a critical value of the* C^2 *map* $\eta \mapsto S(x, \eta)$.

This is Proposition 3.7 p. 71. The minmax defines almost everywhere a section of the projection $T^*M \to M$ restricted to L ("graph selector"):

Theorem 1.11 (Sikorav, Chaperon [21, 53])**.** *Suppose* $L \subset T^*M$ *admits a generating family quadratic at infinity* S*, then* R_S *is a Lipschitz function and there exists an open set* $\Omega \subset M$ *with full measure such that for* $x \in \Omega$*,*

$$(x, dR_S(x)) \in L.$$

1.2 The case $M = \mathbb{R}^d$

In the rest of the chapter, we will take the manifold M to be $\mathbb{R}^d$, in which case the generating families are constructed explicitly. For a general manifold, one can embed it into some $\mathbb{R}^d$ and use the trick of Chekanov [23, 13] to obtain generating families from those in $\mathbb{R}^d$.

1.2.1 Construction of generating functions and phases

Hypotheses and notation In the following, we equip $\mathbb{R}^k$ with the Euclidien ℓ^2 norm $|\cdot|$, and matrices in $\mathbb{R}^k$ with the associated operator norm. We denote by $\mathrm{Lip}(f)$ the Lipschitz constant of a function f and by $\pi : T^*\mathbb{R}^d \to \mathbb{R}^d$ the canonical projection $\pi(x, y) = x$.

We denote by $H : [0, T] \times T^*\mathbb{R}^d \to \mathbb{R}$ a C^2 Hamiltonian satisfying

$$c_H := \sup |D^2 H_t(x, y)| < \infty \tag{1.1}$$

and by X_{H_t} the associated time-depending Hamiltonian vector field [2]. By the general theory of differential equations, as $c_H = \max_t \mathrm{Lip}(DH_t) = \max_t \mathrm{Lip}(X_{H_t})$, the Hamiltonian transformation $\varphi_H^{s,t}$ obtained by integrating X_{H_τ} from $\tau = s$ to $\tau = t$ is a well-defined diffeomorphism for all $(s, t) \in [0, T]$ and

$$\mathrm{Lip}(\varphi_s^t - Id) \le e^{c_H |t-s|} - 1 :$$

see, e.g., Théorème 7.2.1 in [22]. For simplicity, we sometimes write $\varphi_s^t = (X_s^t, Y_s^t) := \varphi_H^{s,t}$ without mentioning H.

We will be mostly interested in the special case where H has compact support, and consider the Lagrangian submanifolds of $T^*\mathbb{R}^d$ which are Hamiltonianly isotopic to the zero section:

$$\mathcal{L} := \{L = \varphi(dv), \quad v \in C^2 \cap C^{\mathrm{Lip}}(\mathbb{R}^d),\, \varphi \in Ham_c(T^*\mathbb{R}^d)\};$$

2. We use the convention of sign that $X_H = (\partial_p H, -\partial_q H)$.

here $C^{\mathrm{Lip}}(\mathbb{R}^d)$ denotes the space of globally Lipschitz functions and

$$\begin{aligned} dv &:= \{(x, dv(x)),\, x \in \mathbb{R}^d\} \subset T^*\mathbb{R}^d \\ Ham_c(T^*\mathbb{R}^d) &= \{\varphi = \varphi_H, \quad H \in C^2_c([0,1] \times T^*\mathbb{R}^d)\}\,^3 \end{aligned}$$

where $\varphi_H = \varphi_H^{0,1}$ is the endpoint of the isotopy ("Hamiltonian flow") defined by H.

Lemma 1.12. *If*

$$\delta_H := c_H^{-1} \ln 2$$

then, for $|s-t| < \delta_H$, the map

$$\alpha_s^t : (x,y) \mapsto (X_s^t(x,y), y)$$

is a diffeomorphism.

Proof. As $e^{c_H \delta_H} = 2$ by definition, we have $\mathrm{Lip}(\alpha_s^t - Id) \leq \mathrm{Lip}(\varphi_s^t - Id) < 1$ for $|t-s| < \delta_H$; it follows that α_s^t is a diffeomorphism and that its inverse is Lipschitzian with

$$\begin{aligned} \mathrm{Lip}\left((\alpha_s^t)^{-1}\right) &= \mathrm{Lip}\left(\left(Id - (Id - \alpha_s^t)\right)^{-1}\right) \leq \left(1 - \mathrm{Lip}(\alpha_s^t - Id)\right)^{-1} \\ &\leq \left(1 - (e^{c_H|t-s|} - 1)\right)^{-1} = (2 - e^{c_H|t-s|})^{-1} : \end{aligned}$$

see for example Théorème 6.1.2 in [22]. □

Definition 1.13. A diffeomorphism $\varphi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ admits a *generating function* ϕ, if $\phi : T^*\mathbb{R}^d \to \mathbb{R}$ is of class C^2, such that $((x,y),(X,Y)) \in \mathrm{Graph}(\varphi)$ if and only if

$$\begin{cases} x = X + \partial_y \phi(X,y) \\ Y = y + \partial_X \phi(X,y). \end{cases}$$

This can be interpreted as follows: the isomorphism

$$\begin{aligned} I : \quad T^*\mathbb{R}^d \times T^*\mathbb{R}^d &\to T^*(T^*\mathbb{R}^d) \\ (x,y,X,Y) &\mapsto (X, y, Y-y, x-X) \end{aligned}$$

is symplectic if $T^*\mathbb{R}^d$ is equipped with the standard symplectic form $\omega = dx \wedge dy$ and $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$ with the symplectic form $(-\omega) \oplus \omega = dX \wedge dY - dx \wedge dy$; this symplectic isomorphism I sends the diagonal of the space $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$ to the zero section of the cotangent space $T^*(T^*\mathbb{R}^d)$ and $\mathrm{Graph}(\varphi)$ to $\mathrm{Graph}(d\phi)$.

Hence, if it exists, the generating function ϕ is unique up to the addition of a constant.

Proposition 1.14. *For $|t-s| < \delta_H$, φ_s^t admits the generating function*

$$\phi_s^t(X,y) = \int_s^t \Big((Y_s^\tau - y)\dot{X}_s^\tau - H(\tau, X_s^\tau, Y_s^\tau)\Big) d\tau \tag{1.2}$$

where $(X_s^\tau(X,y), Y_s^\tau(X,y)) = \varphi_s^\tau \circ (\alpha_s^t)^{-1}(X,y)$ and the dot denotes the derivative with respect to τ.

Proof. If $\lambda = ydx$ denotes the Liouville form of $T^*\mathbb{R}^d$ and V_τ the Hamiltonian vector field of H_τ, we have $\omega = -d\lambda$, hence $(d\lambda)V_\tau = -\omega V_\tau = -dH_\tau$ and therefore

$$\frac{d}{d\tau}(\varphi_s^\tau)^*\lambda = (\varphi_s^\tau)^*\mathcal{L}_{V_\tau}\lambda = (\varphi_s^t)^*\big((d\lambda)V_\tau + d(\lambda V_\tau)\big) = (\varphi_s^\tau)^* d(\lambda V_\tau - H_\tau),$$

yielding

$$\begin{aligned} Y_s^t dX_s^t &= (\varphi_s^t)^*\lambda = \lambda + \int_s^t \frac{d}{d\tau}(\varphi_s^\tau)^*\lambda\, d\tau = \lambda + d\int_s^t (\varphi_s^\tau)^*(\lambda V_\tau - H_\tau)\, d\tau \\ &= y\, dx + d\int_s^t \Big(Y_s^\tau \dot{X}_s^\tau - H(\tau, X_s^\tau, Y_s^\tau)\Big)d\tau = y\, dx + d\big(y(X_s^t - x)\big) + d\phi_s^t, \end{aligned}$$

that is $d\phi_s^t = (Y_s^t - y)dX_s^t + (x - X_s^t)dy$ where $X_s^t \equiv X$. □

Remark 1.15. The fact that φ_s^t admits a generating function follows from Lemma 1.12: indeed, φ_s^t is symplectic if and only if the 1-form $(Y_s^t - y)dX_s^t + (x - X_s^t)dy$ is closed, i.e. exact. The novelty in Proposition 1.14 is the formula for ϕ_s^t.

The isomorphism I provides a global symplectic tubular neighbourhood of the diagonal in $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$ for the symplectic form $(-\omega)\oplus\omega$. By Weinstein's (local) symplectic tubular neighborhood theorem, for each symplectic manifold (M, ω), one can identify similarly a neighborhood of the identity in $Ham_c(M)$ with a neighborhood of zero in the space of exact 1-forms on M. What will be missing in this general case is the existence of a "generating function" for *any* $\varphi \in Ham_c(M)$, obtained as follows if $M = T^*\mathbb{R}^d$:

Lemma 1.16. *For the generating function ϕ_s^t defined in (1.2), we have*

$$\partial_s \phi_s^t(X, y) = H(s, x, y), \quad \partial_t \phi_s^t(X, y) = -H(t, X, Y)$$

where $(X, Y) = \varphi_s^t(x, y)$.

Proof. Derive (1.2) on both sides, we have

$$\begin{aligned} \partial_s \phi_s^t(X, y) &= H(s, x, y) + \int_s^t \Big(\frac{d}{ds}Y_s^\tau \frac{d}{d\tau}X_s^\tau + (Y_s^\tau - y)\frac{d}{ds}\frac{d}{d\tau}X_s^\tau + \frac{d}{d\tau}Y_s^\tau \frac{d}{ds}X_s^\tau - \frac{d}{d\tau}X_s^\tau \frac{d}{ds}Y_s^\tau\Big)d\tau \\ &= H(s, x, y) + \int_s^t (Y_s^\tau - y)\frac{d}{ds}\frac{d}{d\tau}X_s^\tau d\tau + Y_s^\tau \frac{d}{ds}X_s^\tau|_s^t - \int_s^t Y_s^\tau \frac{d}{d\tau}\frac{d}{ds}X_s^\tau d\tau \\ &= H(s, x, y) \end{aligned}$$

where we have used $\partial_1 H_\tau(X_s^\tau, Y_s^\tau) = -\dot{Y}_s^\tau$, $\partial_2 H_\tau(X_s^\tau, Y_s^\tau) = \dot{X}_s^\tau$, and $X_s^t \equiv X$. Similarly, we have

$$\partial_t \phi_s^t(X, y) = -H(t, X, Y)$$

□

Proposition 1.17 (Composition formula 1 [19, 20])**.** *If ϕ_1 and ϕ_2 are generating functions for two diffeomorphisms φ_1, $\varphi_2 : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ respectively, then $\varphi_2 \circ \varphi_1$ admits the generating function*[4]

$$\phi(x_2, y_0; (x_1, y_1)) = \phi_1(x_1, y_0) + \phi_2(x_2, y_1) + (x_2 - x_1)(y_1 - y_0)$$

in the sense that $((x_0, y_0), (x_2, y_2)) \in \mathrm{Graph}(\varphi_2 \circ \varphi_1)$ *if and only if there exists $z = (x_1, y_1)$ such that*

$$\begin{cases} x_0 = x_2 + \partial_{y_0}\phi(x_2, y_0; z) \\ y_2 = y_0 + \partial_{x_2}\phi(x_2, y_0; z) \\ 0 = \partial_z \phi(x_2, y_0; z). \end{cases}$$

4. Better called generating phase or generating family.

More precisely, 0 *is a regular value of* $\partial_z\phi$ *and the map*

$$(x_2, y_0; z) \mapsto \Big((x_2 + \partial_{y_0}\phi(x_2, y_0; z), y_0), (x_2, y_0 + \partial_{x_2}\phi(x_2, y_0; z)\Big)$$

is a diffeomorphism of the submanifold $\Sigma_\phi := \partial_z\phi^{-1}(0)$ *onto* $\mathrm{Graph}(\varphi_2 \circ \varphi_1)$.

The proof is easy. Proposition 1.17 is a generalization [18] of the so-called broken geodesics method: in the situation of Proposition 1.14, if $\varphi_i = \varphi_{t_{i-1}}^{t_i}$ with $0 \le t_i - t_{i-1} < \delta_H$ for $i = 1, 2$, the equation $\partial_z\phi(x_2, y_0; z) = 0$ means that the arc $[t_1, t_2] \ni t \mapsto \varphi_{t_1}^t \circ (\alpha_{t_1}^{t_2})^{-1}(x_2, y_1)$ begins at the endpoint of the arc $[t_0, t_1] \ni t \mapsto \varphi_{t_0}^t \circ (\alpha_{t_0}^{t_1})^{-1}(x_1, y_0)$.

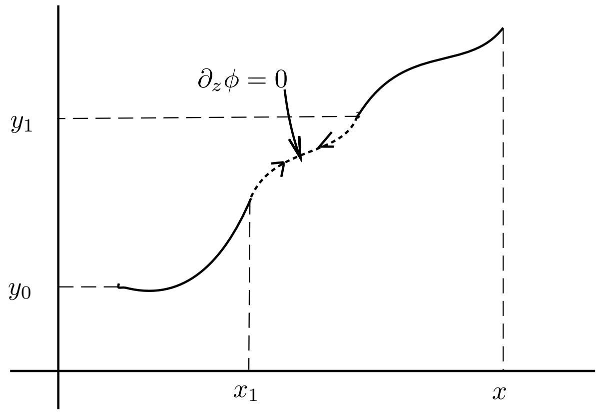


Figure 1.1: connecting of characteristics

Proposition 1.18 (Composition formula 2 [57]). *If a Lagrangian submanifold* $L_0 \subset T^*\mathbb{R}^d$ *admits a generating family* $S_0 : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$, *then for* $|t - s| < \delta_H$, *the Lagrangian submanifold* $\varphi_s^t(L_0)$ *has the generating family*

$$S(x, (\xi, x_0, y_0)) = S_0(x_0, \xi) + \phi_s^t(x, y_0) + xy_0 - x_0y_0 \tag{1.3}$$

Again, the proof is straightforward.

Corollary 1.19. *For each subdivision* $0 \le s = t_0 < t_1 \cdots < t_N = t \le T$ *satisfying* $|t_i - t_{i+1}| < \delta_H$, *if* $\phi_H^{t_i,t_{i+1}}$ *is the generating function of* $\varphi_H^{t_i,t_{i+1}}$ *defined in Proposition 1.14, we have the following for each* C^2 *function* $v : \mathbb{R}^d \to \mathbb{R}$:

i) A generating family $S : \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \to \mathbb{R}$ *of the Lagrangian submanifold* $\varphi_H^{s,t}(dv)$ *is*

$$S(x, \eta) = v(x_0) + \sum_{0\le i<N} \phi_H^{t_i,t_{i+1}}(x_{i+1}, y_i) + \sum_{0\le i<N}(x_{i+1} - x_i)y_i, \tag{1.4}$$

where $x_N := x$, $\eta = \big((x_i, y_i)\big)_{0\le i<N}$.

ii) One defines a C^2 *family* $S : [s, t] \times \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \to \mathbb{R}$ *such that each* $S_\tau := S(\tau, \cdot)$ *is a generating family for* $\varphi_H^{s,\tau}(dv)$ *as follows: let* $\tau_j = s + (\tau - s)\frac{t_j - s}{t - s}$,

$$S(\tau, x, \eta) = v(x_0) + \sum_{0\le i<N} \phi_H^{\tau_i,\tau_{i+1}}(x_{i+1}, y_i) + \sum_{0\le i<N}(x_{i+1} - x_i)y_i \tag{1.5}$$

iii) For each critical point η of $S(\tau, x; \cdot)$, the corresponding critical value is

$$S_\tau(x;\eta) = v(x_0) + \int_s^\tau \Big(Y_s^\sigma \dot{X}_s^\sigma - H(\sigma, X_s^\sigma, Y_s^\sigma)\Big)\, d\sigma,$$

where $X_s^\sigma := X_s^\sigma(x_0, dv(x_0))$ and $Y_s^\sigma := Y_s^\sigma(x_0, dv(x_0))$. Hence, the critical values of $S(\tau, x; \cdot)$ are the real numbers

$$v(X_\tau^s(z)) + \int_s^\tau \Big(Y_\tau^\sigma(z)\dot{X}_\tau^\sigma(z) - H(\sigma, X_\tau^\sigma(z), Y_\tau^\sigma(z))\Big)\, d\sigma \tag{1.6}$$

with $z := (x, y)$, $y \in \pi^{-1}(x) \cap \varphi_H^{s,\tau}(dv)$.

Proof. i) As the Hamiltonian flow is a "two-parameter groupoid", we have that

$$\varphi_H^{s,t} = \varphi_H^{t_0,t_N} = \varphi_H^{t_{N-1},t_N} \circ \cdots \circ \varphi_H^{t_0,t_1};$$

hence, if $|t_{i+1} - t_i| < \delta_H$ for all i, it follows from the composition formula in Proposition 1.18 that formula (1.4) does define a generating family for $\varphi_H^{s,t}(dv)$.

ii) is clear.

iii) is proved by inspection (and very important). □

1.2.2 Generating functions (families) quadratic at infinity

In Proposition 1.17, when φ_1 and φ_2 have compact support [5], so do [6] ϕ_1 and ϕ_2; if we make the change of variables $\xi := x_2 - x_1$, $\eta := y_1 - y_0$, the generating phase ϕ writes $\psi(x_2, y_0; \xi, \eta) := \phi_1(x_2 - \xi, y_0) + \phi_2(x_2, y_0 + \eta) + \eta\xi$, which is again a generating phase of $\varphi_2 \circ \varphi_1$; the difference $\psi(x_2, y_0; \xi, \eta) - \eta\xi$ does not have compact support in general but its differential is bounded, which makes it quadratic at infinity in a sense good enough for most applications [19, 20].

We now give a weaker definition of "quadratic at infinity", which will include such cases and take into account the non compactness of the base manifold.

Definition 1.20. A family $S : M \times \mathbb{R}^k \to \mathbb{R}$ is called *(almost) quadratic at infinity* if there exists a nondegenerate quadratic form $Q : \mathbb{R}^k \to \mathbb{R}$ such that, for any compact subset $K \subset M$, the restriction $S|_{K\times\mathbb{R}^k}$, modulo a fiberwise diffeomorphism, equals Q off a compact set.

The next Propostion gives a criterion for a family to be (almost)quadratic at infinity, it extends the result in [64, 60].

Proposition 1.21. *Suppose a family $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is of the form*

$$S(x, \eta) = \psi(x, \eta) + Q(\eta) := \ell(x, \eta) + \psi_1(x, \eta) + Q(\eta)$$

where $Q(\eta) = \frac{1}{2}\eta^T B\eta$ is a nondegenerate quadratic form, ℓ is a C^2 function such that $\partial_\eta \ell$ is bounded in $K \times \mathbb{R}^k$ for each compact $K \subset \mathbb{R}^d$, and ψ_1 is C^2 with

$$c := \sup |\partial_\eta^2 \psi_1(x, \eta)| < |B^{-1}|^{-1}. \tag{1.7}$$

Then S is quadratic at infinity.

5. Meaning that they equal the identity off a compact subset.
6. In the usual sense, up to the addition of a constant.

Proof. Given any compact set $K \subset \mathbb{R}^d$, we restrict ourselves to $x \in K$. Consider a smooth function $\theta : \mathbb{R}^+ \to [0,1]$ with $\theta = 1$ on $[0,a]$, $\theta = 0$ on $[a',\infty)$, and $0 \leq \theta'(s) \leq s^{-1}\epsilon$, where $\epsilon > 0$ will be chosen small enough. If

$$S_K(x,\eta) = \psi_K(x,\eta) + Q(\eta) := \theta(|\eta|)\psi(x,\eta) + Q(\eta), \quad x \in K,$$

we claim that $S|_{K\times\mathbb{R}^k}$ and S_K will be equivalent by a fiberwise diffeomorphism: setting $S_t := tS + (1-t)S_K$, i.e.

$$S_t(x,\eta) = \big(t + (1-t)\theta(|\eta|)\big)\psi(x,\eta) + Q(\eta)$$

for $0 \leq t \leq 1$, we will find a fiberwise isotopy $\Phi_t(x,\eta) = (x, \phi_t(x,\eta))$ such that

$$S_t \circ \Phi_t = S_K \text{ for all } t \in [0,1] \tag{$*$}$$

and therefore $S \circ \Phi_1 = S_K$, as required.

If $(0, X_t)$ denotes the infinitesimal generator of Φ_t, then $(*)$ is equivalent to

$$\partial_\eta S_t(x,\eta)\cdot X_t(x,\eta) + \big(1-\theta(|\eta|)\big)\psi(x,\eta) = 0 \text{ for all } t \in [0,1]; \tag{$**$}$$

note that for $|\eta| \leq a$, as $S = S_K = S_t$, we can take $\Phi_t = Id$, i.e. $X_t = 0$.

Our hypotheses imply that there are constants $b_1(K), b_2(K), b_3(K) \geq 0$ such that

$$\begin{aligned}
|\psi(x,\eta)| &= |\psi(x,0) + \partial_\eta\psi_1(x,0)\eta + \textstyle\int_0^1 (1-t)\partial_\eta^2\psi_1(x,t\eta)\eta^2\,dt + \int_0^1 \partial_\eta\ell(x,t\eta)\eta\,dt| \\
&\leq b_1 + b_2|\eta| + \tfrac{c}{2}|\eta|^2 \\
|\partial_\eta\psi(x,\eta)| &= |\partial_\eta\psi_1(x,0) + \textstyle\int_0^1 \partial_\eta^2\psi_1(x,t\eta)\eta\,dt + \partial_\eta\ell(x,\eta)| \\
&\leq b_3 + c|\eta|,
\end{aligned}$$

hence

$$\begin{aligned}
|\partial_\eta(S_t - Q)(x,\eta)| &= \Big|\partial_\eta\Big(\big(t + (1-t)\theta(|\eta|)\big)\psi(x,\eta)\Big)\Big| \\
&\leq |\partial_\eta\psi(x,\eta)| + |\psi(x,\eta)\partial_\eta\theta(|\eta|)| \\
&\leq b_3 + c|\eta| + \tfrac{\epsilon}{|\eta|}(b_1 + b_2|\eta| + \tfrac{c}{2}|\eta|^2) \\
&\leq (\tfrac{b_1}{|\eta|} + b_2)\epsilon + b_3 + (1+\tfrac{\epsilon}{2})c|\eta|.
\end{aligned}$$

As $|DQ(\eta)| = |B\eta| \geq |B^{-1}|^{-1}|\eta|$, this yields

$$|\partial_\eta S_t(x,\eta)| \geq |DQ| - |\partial_\eta(S_t - Q)| \geq \big(|B^{-1}|^{-1} - (1+\tfrac{\epsilon}{2})c\big)|\eta| - (\tfrac{b_1}{|\eta|} - b_2)\epsilon - b_3;$$

by (1.7), we have $|B^{-1}|^{-1} - (1+\frac{\epsilon}{2})c > 0$ for $\epsilon > 0$ small enough; if this is the case, for $0 < c' < |B^{-1}|^{-1} - (1+\frac{\epsilon}{2})c$, we have $\big(|B^{-1}|^{-1} - (1+\frac{\epsilon}{2})c\big)|\eta| - (\frac{b_1}{|\eta|} - b_2)\epsilon - b_3 \geq c'|\eta|$ when $|\eta|$ is large enough. Given such a constant c', if we now take a large enough and let

$$X_t(x,\eta) = \begin{cases} \frac{(\theta-1)\psi(x,\eta)}{|\partial_\eta S_t(x,\eta)|^2}\partial_\eta S_t(x,\eta) & \text{for } |\eta| \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

it satisfies $(**)$ and is integrable since there are positive constants d_1, d_2 such that

$$|X_t(x,\eta)| \leq \frac{|\psi(x,\eta)|}{|\partial_\eta S_t(x,\eta)|} \leq \frac{b_1 + b_2|\eta| + \frac{c}{2}|\eta|^2}{c'|\eta|} \leq d_1 + d_2|\eta|$$

for $|\eta| \geq a$. □

Remark 1.22. If S is a generating family for a Lagrangian submanifold L, then S_K generates $L|_K = \{(x,p) \in L | x \in K\}$ since there is no critical points for S_K outside $|\eta| \leq a$ if we choose a large enough.

Note that a necessary condition for L to admit a GFQI in our new sense is that, for any K, the intersection $L \cap \pi^{-1}(K)$ be compact and *nonempty*: indeed, a function on $\mathbb{R}^k$ equal to a nondegenerate quadratic form off a compact set must have critical points.

It follows that there does not always exist a GFQI for $L = \varphi_H^{s,t}(dv)$ if H is not compactly supported, even when it satisfies (1.1) and v has as little growth at infinity as possible:

Example 1.23. If the Hamiltonian $H \in C^2([0,T] \times T^*\mathbb{R})$ is given by $H(t,x,y) = x^2 + y^2$, then $\varphi_H^{0,t}(x,y) = (x \cos 2t - y \sin 2t, y \cos 2t + x \sin 2t)$; if $v = 0$, it follows that

$$L := \varphi_H^{0,\pi/4}(dv) = \{0\} \times \mathbb{R}$$

has empty intersection with $\pi^{-1}(x) = \{x\} \times \mathbb{R}$ for $x \neq 0$ and noncompact intersection with $\pi^{-1}(0)$, which prevents L from admitting a GFQI.

When (1.1) is satisfied, however, the Lagrangian $L = \varphi_H^{s,t}(dv)$ does admit a GFQI for small $|s-t|$. Indeed, as $\varphi_H^{s,t}$ is close to the identity, the generating function $\phi_H^{s,t}$ is "small" compared to the quadratic form, hence Proposition 1.21 applies:

Corollary 1.24. *If* (1.1) *is satisfied then, for each Lipschitzian C^2 function $v : \mathbb{R}^d \to \mathbb{R}$, there exists a constant α such that for $|t-s| < \alpha$,*

$$S(x; x_0, y_0) = v(x_0) + \phi_H^{s,t}(x, y_0) + xy_0 - x_0 y_0$$

is a GFQI for $L = \varphi_H^{s,t}(dv)$.

Proof. This follows from Proposition 1.21 with $Q(x_0, y_0) := -x_0 y_0$, $\ell(x; x_0, y_0) := v(x_0) + xy_0$ and $\psi_1 := \phi_H^{s,t}$. Indeed, Since $|Q^{-1}| = |Q| = 1$, it is enough to prove that $|D^2 \phi_H^{s,t}| < 1$ for $|s-t| < \alpha$. Now, as

$$\partial_X \phi_H^{s,t} \circ \alpha_s^t(x,y) = Y_s^t(x,y) - y, \quad \partial_y \phi_H^{s,t} \circ \alpha_s^t(x,y) = x - X_s^t(x,y),$$

we have

$$|D^2 \phi_H^{s,t}| \leq \operatorname{Lip}\left((\alpha_s^t)^{-1}\right) \operatorname{Lip}(\varphi_H^{s,t} - Id) \leq \frac{e^{c_H|t-s|} - 1}{2 - e^{c_H|t-s|}}\,, \tag{1.8}$$

hence we can take $\alpha = c_H^{-1} \log(3/2)$. □

Remark 1.25. It is essential that v be Lipschitzian: indeed, if $d = 1$, $H(t,x,y) = \frac{1}{2}y^2$ and $v(x) = \frac{1}{3}x^3$, then $\varphi_H^{0,t}(x,y) = (x + ty, y)$ and therefore $\varphi_H^{0,t}(dv) = \{(x + tx^2, x^2)\}$, whose image under the projection π is a half-line for $t \neq 0$.

Corollary 1.26. *If H has compact support, the generating phases constructed in Corollary 1.19 are quadratic at infinity when the C^2 function v is Lipschitzian.*

Proof. Each $\phi_H^{t_i,t_{i+1}}$ has compact support and therefore bounded derivatives. Hence we can apply Lemma 1.21 with $\psi_1 = 0$, $\ell(x;\eta) = v(x_0) + xy_{N-1} + \sum_{0 \leq i < N} \phi_H^{t_i,t_{i+1}}(x_{i+1}, y_i)$ and $Q(\eta) := -x_{N-1}y_{N-1} + \sum_{0 \leq i < N-1}(x_{i+1} - x_i)y_i$. □

As the main ingredient in this construction is the Hamiltonian flow, what matters essentially over a given compact subset of $\mathbb{R}^d$ is the region swept by the Hamiltonian flow; this is the idea of what is called the *property of finite propagation speed* in [15], Appendix A:

Proposition 1.27 ([15]). *Let $[s,t] \subset [0,T]$ and $L = \varphi_H^{s,t}(dv)$. If for any compact subset $K \subset \mathbb{R}^d$, the set*

$$\mathcal{U}_K := \bigcup_{\tau\in[s,t]} \{\tau\} \times \left\{\varphi_H^{s,\tau}\left(\varphi_H^{t,s}(\pi^{-1}(K)) \cap dv\right)\right\},$$

is compact, then L admits GFQI's in the sense that each $L|_K := L \cap \pi^{-1}(K)$ has a GFQI.

Proof. For any K, let $\tilde{H} = \chi H$, where χ is a compactly supported smooth function on $[0,T] \times T^*\mathbb{R}^d$ equal to 1 in a neighbourhood of $\mathcal{U}_K$. Then formula (1.4) with $H := \tilde{H}$ gives a GFQI $S_{\tilde{H}}$ for $L|_K = \varphi_H^{s,t}(\pi^{-1}(K) \cap dv)$. □

Remark 1.28. One can also truncate v, as the effective region for v is $\pi(\varphi_H^{t,s}(\pi^{-1}(K))$. This may help to localize the minmax.

Condition (1.1) is not required here, provided H is C^2 and such that $\varphi_H^{s,t}$ is defined for all $s, t \in [0,T]$.

Lemma 1.29. *If two families S and S' are quadratic at infinity with $|S - S'|_{C^0} < \infty$, then the associated minimax functions satisfy*

$$|R_S(x) - R_{S'}(x)| \leq |S - S'|_{C^0}.$$

Proof. If $S \leq S'$, then by definition $R_S(x) \leq R_{S'}(x)$. Hence, in general, the inequality $S \leq S' + |S - S'|_{C^0}$ yields $R_S(x) \leq R_{S'}(x) + |S - S'|_{C^0}$. We conclude by exchanging S and S'. □

Proposition 1.30 ([15]). *Under the hypotheses of Proposition 1.27 and with the notation of its proof, the Lagrangian submanifold L admits a minmax selector, given by*

$$R(x) = \inf\max S_{\tilde{H}}(x,\eta), \quad \textit{if } x \in K \subset \mathbb{R}^d$$

and independent of the truncation $\tilde{H}$ and the subdivision of $[s,t]$ used to define $S_{\tilde{H}}$.

Proof. Let $\tilde{H}$ and $\tilde{H}'$ be two truncations for H on $\mathcal{U}_K$ as in the proof of Proposition 1.27. Let $H^\mu = \mu\tilde{H} + (1-\mu)\tilde{H}'$, $\mu \in [0,1]$; as the constant c_{H^μ} of (1.1) is uniformly bounded, one can find a subdivision $s = t_0 < t_1 \cdots < t_N = t$ satisfying $|t_i - t_{i+1}| < \delta_{H_\mu}$ for all μ (see Lemma 1.12); if S_μ denotes the corresponding GFQI of $L|_K = L \cap \pi^{-1}(K)$ for $0 \leq \mu \leq 1$ then, by Lemma 1.29, as S_μ depends continuously on μ, so does the minmax $R_{S_\mu}(x)$ for $x \in \pi(L)$.

On the other hand, $R_{S_\mu}(x)$ is a critical value of the map $\eta \mapsto S_\mu(x,\eta)$, and, by (1.6), *the set of all such critical values is independent of μ and the subdivision, and depends only on $\mathcal{U}_K$*; as it has measure zero by Sard's Theorem, $R_{S_\mu}(x)$ is constant for $\mu \in [0,1]$.

The fact that the critical value $R_S(x)$ itself does not depend on the subdivision is established in Lemma 2.6. □

Example 1.31. If the base manifold $M = \mathbb{T}^d$, taking its universal covering $\mathbb{R}^d$, we can consider $v : \mathbb{R}^d \to \mathbb{R}$ a periodic function and $H : \mathbb{R} \times T^*\mathbb{R}^d \to \mathbb{R}$ periodic in x. Then in order that $L = \varphi_H^{s,t}(dv)$ admits a GFQI, it is enough to require that the flow $\varphi_H^{s,\tau}$ is well-defined for $\tau \in [s,t]$. Indeed, since dv is compact, $\bigcup_{\tau\in[s,t]}\{\tau\} \times \varphi_H^{s,\tau}(dv)$ is compact, hence the condition of finite propagation speed is satisfied automatically.

Proposition 1.32. *Suppose H satisfies* (1.1) *and*

$$C_H := \sup \frac{|\partial_x H(t,x,y)|}{1+|y|} < \infty\,; \tag{1.9}$$

then, for $v \in C^2 \cap C^{\mathrm{Lip}}(\mathbb{R}^d)$, the submanifold $L = \varphi_H^{s,t}(dv)$ admits GFQI's in the sense of Proposition 1.27.

Proof. With the notation of Proposition 1.27, let $(x(\tau), y(\tau)) := \varphi_s^\tau\Big(x(s), dv(x(s))\Big)$ for $\tau \in [s,t]$. As

$$y(\tau) = y(s) - \int_s^\tau \partial_x H(\sigma, x(\sigma), y(\sigma)) d\sigma, \quad y(s) = dv(x(s)),$$

it follows from (1.9) that the function $f(\tau) := \int_s^\tau |y(\tau)| d\tau$ satisfies

$$f'(\tau) = |y(\tau)| \leq |dv(x(s))| + C_H(\tau - s + f(\tau)) \leq \mathrm{Lip}(v) + C_H(t - s + f(\tau)). \tag{1.10}$$

If $C_H = 0$, this writes $|y(\tau)| \leq \mathrm{Lip}(v)$; otherwise, (1.10) can be written as

$$\frac{d}{d\tau}(f(\tau)e^{-C_H\tau}) \leq e^{-C_H\tau}\big(\mathrm{Lip}(v) + C_H(t-s)\big),$$

hence

$$f(\tau) \leq \big(\mathrm{Lip}(v) + C_H(t-s)\big)\frac{1 - e^{C_H(s-\tau)}}{C_H} \leq \big(\mathrm{Lip}(v) + C_H(t-s)\big)\frac{1 - e^{C_H(s-t)}}{C_H};$$

therefore, by (1.10), the set of all $|y(\tau)|$ with $\tau \in [s,t]$ and $\big(x(s), y(s)\big) \in dv$ is bounded. It follows that

$$|\partial_y H_\tau(x(\tau), y(\tau))| = |\partial_y(H(0, y(\tau)) + \int_0^1 \partial_x\partial_y H_\tau(ux(\tau), y(\tau))x(\tau)\, du)| \leq c + c_H|x(\tau)|,$$

and the same argument as before shows that the set of all

$$x(\tau) = x(t) - \int_\tau^t \partial_y H_\tau(x(\tau), y(\tau)) d\tau$$

with $\tau \in [s,t]$ and $x(t) \in K$ is bounded. □

Remark 1.33. One can also use the hypotheses

$$|\partial_y H| \leq C'_H(1 + |x|), \quad |\partial_x H| \leq C_H(1 + |y|),$$

a classical condition for the existence and uniqueness of viscosity solutions in $\mathbb{R}^d$, see [29].

1.3 Generalized generating families and minmax in the Lipschitz cases

Already if $d = 1$, $H(t,x,y) = \frac{1}{2}y^2$ and $v(x) = \arctan x$, the Lagrangian submanifold $\varphi_H^{0,t}(dv) = \Big\{\Big(x + \frac{t}{1+x^2}, \frac{1}{1+x^2}\Big) : x \in \mathbb{R}\Big\}$ is not the graph of a function for $t > 0$ large enough, and the minimax of its generating phase $S_t(x; x_0, y_0) = \arctan x_0 + \frac{t}{2}y_0^2 + (x - x_0)y_0$ is not a C^1 function, though it is locally Lipschitzian (see Proposition 1.40 herafter).

Hence, in order to to iterate the minmax procedure, one is led to defining the minmax when the Cauchy datum is a Lipschitzian function. We will use Clarke's generalization of the derivatives of C^1 functions in the Lipschitz setting [27], see Appendix A.

Proposition 1.34. *Under the hypothesis* (1.1) *and with the notation of Corollary 1.19, if v is only locally Lipschitzian, the family S given by* (1.4) *generates $L = \varphi_H^{s,t}(\partial v)$ in the sense that*

$$L = \big\{(x, \partial_x S(x;\eta)) | 0 \in \partial_\eta S(x;\eta)\big\}. \tag{1.11}$$

Proof. The equation $0 \in \partial_\eta S(x;\eta)$ means that $y_0 \in \partial v(x_0)$ and $y_{i+1} = \partial_{x_{i+1}} \phi_H^{t_i,t_{i+1}}(x_{i+1}, y_i)$, $x_i = \partial_{y_i} \phi_H^{t_i,t_{i+1}}(x_{i+1}, y_i)$ for $0 \le i < N$,where $x := x_N$ et $\eta = (x_i, y_i)_{0 \le i < N}$. □

However, this definition of a generating family is not invariant by fiberwise diffeomorphism, even by the following very simple (and useful) one:

$$(x; (x_i)_{0\le i<N}, (y_i)_{0\le i<N}) \mapsto (x, (x_{i+1} - x_i, y_i)_{0\le i<N}) =: (x, (\xi_i, y_i)_{0\le i<N});$$

indeed, it transforms the family S given by (1.4) into

$$S'(x; (\xi_i, y_i)_{0\le i<N}) := v\Big(x - \sum \xi_i\Big) + \sum_{0\le i<N} \phi_H^{t_i,t_{i+1}}\Big(x - \sum_{i<j<N} \xi_j, y_i\Big) + \sum_{0\le i<N} \xi_i y_i\,,$$

for which $\partial_x S'(x; (\xi_i, y_i)_{0\le i<N})$ is not a point, but the subset

$$\partial v\Big(x - \sum \xi_i\Big) + \sum_{0\le i<N} \partial_1 \phi_H^{t_i,t_{i+1}}\Big(x - \sum_{i<j<N} \xi_j, y_i\Big).$$

As often, this difficulty is overcome by finding the right definition [7]

Definition 1.35. A Lipschitz family $S : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is called a *generating family* for $L \subset T^*\mathbb{R}^d$ when

$$L = \{(x, y) \in T^*\mathbb{R}^d | \exists \eta \in \mathbb{R}^k : (y, 0) \in \partial S(x, \eta)\}.$$

Lemma 1.36. *This definition of a generating family is invariant by fiberwise C^1 diffeomorphisms.*

Proof. If $\Phi(x, \eta') = (x, \phi(x, \eta'))$ is a fiberwise diffeomorphism of $\mathbb{R}^d \times \mathbb{R}^k$, and $S' := S \circ \Phi$, then the chain rule (see Appendix-A, Lemma A.16) yields

$$\partial S'(x, \eta') = \Big\{ \Big(y + \zeta \tfrac{\partial}{\partial x}\phi(x, \eta'), \zeta \tfrac{\partial}{\partial \eta'}\phi(x, \eta')\Big) \Big| (y, \zeta) \in \partial S(x, \phi(x, \eta')) \Big\}\,;$$

as $\eta' \mapsto \phi(x, \eta')$ is a diffeomorphism, it does follow that the two conditions

$$\exists \eta \in \mathbb{R}^k : (y, 0) \in \partial S(x, \eta) \quad \text{and} \quad \exists \eta' \in \mathbb{R}^k : (y, 0) \in \partial S'(x, \eta')$$

are equivalent. □

Remark 1.37. For the generating family S defined by (1.4) or (1.5), it generates L in the sense of (1.11) since $\partial S(x, \eta) = \partial_x S(x, \eta) \times \partial_\eta S(x, \eta)$. See Example A.6.

We are now ready to consider GFQI's for the elements of

$$\tilde{\mathcal{L}} := \{L = \varphi(\partial v), \quad v \in C^{\mathrm{Lip}}(\mathbb{R}^d),\ \varphi \in Ham_c(T^*\mathbb{R}^d)\} :$$

Proposition 1.38. *If $H : [0, T] \times T^*\mathbb{R}^d \to \mathbb{R}$ is C^2 and has compact support then, for each $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$, the generating family of $L = \varphi_H^{s,t}(\partial v) \in \tilde{\mathcal{L}}$ given by* (1.4)*, namely*

$$S(x; \eta) = v(x_0) + \sum_{0\le i<N} \phi_H^{t_i,t_{i+1}}(x_{i+1}, y_i) + \sum_{0\le i<N} (x_{i+1} - x_i) y_i,$$

7. But this example exhibits one of the features of the Clarke derivative: the relation $(y, 0) \in \partial S'(x, \eta)$ is definitely not equivalent to $y \in \partial_x S'(x, \eta)$, $0 \in \partial_{\xi_i} S'(x, \eta)$ and $0 \in \partial_{y_i} S'(x, \eta)$.

where $x_N := x$, $\eta := \big((x_i, y_i)\big)_{0\le i<N}$, *is " quadratic at infinity" in the following sense: let*

$$Q(\eta) := -x_{N-1}y_{N-1} + \sum_{0\le i<N-1} (x_{i+1} - x_i)y_i,$$

the Lipschitz constant of each $S(x,\cdot) - Q : (T^*\mathbb{R}^d)^N \to \mathbb{R}$ *is bounded, uniformly with respect to* x *on each compact subset of* $\mathbb{R}^d$*.*

Hence, for each compact $K \subset \mathbb{R}^k$*, if* $\theta \in C_c^\infty(\mathbb{R}^d, [0,1])$ *equals* 1 *in a neighbourhood of* 0*, there exists a positive constant* a_K *such that the function*

$$S_K(x;\eta) = \psi_K(x;\eta) + Q(\eta), \text{ where } \psi_K(x;\eta) := \theta\left(\tfrac{\eta}{a_K}\right)(S(x,\eta) - Q(\eta)),\ x \in K \quad (1.12)$$

is a GFQI of $L_K := L \cap \pi^{-1}(K)$*.*

Proof. Denote $\psi(x,\eta) = S(x,\eta) - Q(\eta)$, and $Q(\eta) = \frac{1}{2}\eta^T B\eta$. For a fixed compact subset K, let $c = \max_{x\in K} \mathrm{Lip}(\psi(x,\cdot))$, and assume that $|D\theta| \le 1$. By Lemma A.18,

$$\begin{aligned}
\partial_\eta S_K(x,\eta) &= \partial_\eta(\theta(\frac{\eta}{a_K})\psi(x,\eta) + Q(\eta)) \\
&\subset \frac{1}{a_K} D\theta(\frac{\eta}{a_K})\psi(x,\eta) + \theta(\frac{\eta}{a_K})\partial_\eta\psi(x,\eta) + DQ(\eta),
\end{aligned}$$

By Proposition A.8, we have

$$|\psi(x,\eta)| \le |\psi(x,0)| + |\psi(x,\eta) - \psi(x,0)| \le b + c|\eta|$$

where $b := \max_{x\in K} |\psi(x,0)|$. Hence,

$$|\frac{1}{a_K} D\theta(\frac{\eta}{a_K})\psi(x,\eta) + \theta(\frac{\eta}{a_K})\partial_\eta\psi(x,\eta)| \le \frac{1}{a_K}(b + c|\eta|) + c \le \frac{1}{2}|B^{-1}|^{-1}|\eta| < |DQ(\eta)|$$

when $|\eta| \ge b_K$, for some b_K with a_K, b_K large enough. In addition, we can choose a_K, b_K such that for $|\eta| \le b_K$, $\theta(\frac{\eta}{a_K}) = 1$. Thus $S_K = S$ for $|\eta| \le b_K$ and there are no critical points of S,S_K outside $\{|\eta| \le b_K\}$, from which $L_K = \{(x, \partial_x S_K(x,\eta)) | 0 \in \partial_\eta S_K(x,\eta)\}$. □

In the sequel, *unless otherwise specified, we consider families* S *of the form* (1.4) *or* (1.5) *and families* S_K *of the form* (1.12). The advantage is that the S better generates L in the more geometric sense (1.11), and it helps to express the properties of minmax $R_S(x)$ in a clear and similar way as in the C^2 case.

To study the minmax function R_S for such S, we use the extension of classical results in critical point theory to locally Lipschitz functions described in Appendix A.

Proposition 1.39. *The minmax* $R_S(x)$ *is well-defined and it is a critical value*[8] *of the map* $\eta \mapsto S(x,\eta)$*. For each compact subset* K *of* $\mathbb{R}^d$ *and each truncation* S_K *of* S *of the form* (1.12) *generating* L_K*, we have that* $R_S(x) = R_{S_K}(x)$ *for* $x \in K$*.*

Proof. By Proposition 1.38, $f(\eta) := S(x,\eta) = \psi(\eta) + Q(\eta)$ with ψ Lipschitzian and Q a nondegenerate quadratic form. Hence, f satisfies the P.S. condition (see Appendix A, Example A.11). If $c = R_S(x)$ were not a critical value, the flow φ_V^t of Theorem A.14 in Appendix A would deform the descending cycles in $f^{c+\epsilon}$ into descending cycles in $f^{c-\epsilon}$, hence the contradiction $c = \inf \max_\sigma f \le c - \epsilon$.

To see that $R_S|_K = R_{S_K}$, just notice that every descending cycle σ of $S(x,\cdot)$ or $S_K(x,\cdot)$, $x \in K$, can be deformed into a common descending cycle σ' with $\max S(x,\sigma'(\cdot)) = \max S_K(x,\sigma'(\cdot))$ by using the gradient flow of Q, suitably truncated. □

8. Appendix A, Definition A.10.

Proposition 1.40. *The minmax $R_S(x)$ is a locally Lipschitz function.*

Proof. Let $K \subset \mathbb{R}^d$ be compact. By Proposition 1.39, we have that $R_S|_K = R_{S_K}$, where $S_K : K \times \mathbb{R}^k \to \mathbb{R}$ writes $S(x,\eta) = \psi_K(x,\eta) + Q(\eta)$ with Q a nondegenerate quadratic form and ψ_K a compactly supported Lipschitz function. Given $x, x' \in K$, for all $\epsilon > 0$, there exists a descending cycle $\bar{\sigma}$ such that $\max_{\eta \in \bar{\sigma}} S_K(x,\eta) \leq R_S(x) + \epsilon$; if $\max_{\eta \in \bar{\sigma}} S_K(x',\eta)$ is reached at $\bar{\eta}$, then

$$\begin{aligned} R_S(x') - R_S(x) &\leq S_K(x',\bar{\eta}) - S_K(x,\bar{\eta}) + \epsilon = \psi_K(x',\bar{\eta}) - \psi_K(x,\bar{\eta}) + \epsilon \\ &\leq \operatorname{Lip}(\psi_K)|x - x'| + \epsilon. \end{aligned}$$

If we let $\epsilon \to 0$ and exchange x and x', we obtain

$$|R_S(x) - R_S(x')| \leq \operatorname{Lip}(\psi_K)|x - x'|,$$

which proves our result. □

Proposition 1.41. *The sets $C(x) = \{\eta \,|\, 0 \in \partial_\eta S(x,\eta), S(x,\eta) = R_S(x)\}$ are compact*[9] *and the set-valued map* (“correspondence”) *$x \mapsto C(x)$ is* upper semi-continuous*: for every convergent sequence $(x_k, \eta_k) \to (x,\eta)$ with $\eta_k \in C(x_k)$, one has $\eta \in C(x)$. In other words, the graph $C = \{(x,\eta) \,|\, \eta \in C(x)\}$ of the correspondence is closed.*

Proof. Let $(x_k,\eta_k) \to (x,\eta)$ with $\eta_k \in C(x_k)$; for S defined by (1.4), we have $\partial S = \partial_x S \times \partial_\eta S$. Now $\partial S : (x,\eta) \mapsto \partial_x S \times \partial_\eta S$ is upper semi-continuous (Appendix A, Proposition A.7), the limit $(\partial_x S(x,\eta), 0)$ of the sequence $(\partial_x S(x_k,\eta_k), 0) \in \partial S(x_k,\eta_k)$ belongs to $\partial S(x,\eta)$, hence $0 \in \partial_\eta S(x,\eta)$; as the continuity of S and R_S implies that $S(x_k,\eta_k) \to S(x,\eta)$ and $R_S(x_k) \to R_S(x)$, this proves $\eta \in C(x)$. □

Lemma 1.42. *Given any $\delta > 0$, there exists an $\epsilon > 0$ such that*

$$R_S(x) = \inf_{\sigma \in \Sigma_\epsilon} \max_{\sigma \cap C_\delta(x)} S(x,\eta)$$

where $\Sigma_\epsilon = \{\sigma \,|\, \max_\sigma S(x,\eta) \leq R_S(x) + \epsilon\}$ and $C_\delta(x) = B_\delta(C(x))$ denotes the δ-neighborhood of the critical set $C(x)$.

Proof. This is a direct consequence of the deformation lemma (Appendix A, Theorem A.15) for $S_x := S(x,\cdot)$: for $\delta > 0$, and $c = R_S(x)$, there exist $\epsilon > 0$ and V such that $\varphi_V^1(S_x^{c+\epsilon} \setminus C_\delta(x)) \subset S_x^{c-\epsilon}$. In particular, we remark that for $\sigma \in \Sigma_\epsilon$, the intersection $\sigma \cap C_\delta(x)$ is non vide, otherwise, the flow φ_V^1 may take σ to a descending cycle $\sigma' = \varphi_V^1(\sigma)$ such that $\max_{\eta \in \sigma'} S_x(\eta) \leq R_S(x) - \epsilon$, contradiction with the definition of minmax. □

Remark 1.43. When S is C^2, the S_x's are generically Morse functions: indeed, S_x is Morse if and only if x is a regular value of the projection $\pi : L \to M$, $(x,p) \mapsto x$, whose regular values, by Sard's theorem and the compactness of $\operatorname{Crit}(S_x)$, form an open set of full measure. In this case, S_x^c is indeed a deformation retract of $S_x^{c+\epsilon}$ for $\epsilon > 0$ small enough, hence inf max deserves its name “minmax”, that is, there exists a descending cycle σ such that, $R_S(x) = \max_\sigma S(x,\eta) = \max_{\sigma \cap C(x)} S(x,\eta)$.

Proposition 1.44. *The generalized derivative of R_S satisfies*

$$\partial R_S(x) \subset \operatorname{co}\{\partial_x S(x,\eta) \,|\, \eta \in C(x)\} \tag{1.13}$$

9. See Appendix A, Example A.11 and Proposition A.12.

Proof. First, we claim that, if R_S is differentiable at $\bar{x}$, then

$$dR_S(\bar{x}) \subset \mathrm{co}\{\partial_x S(\bar{x},\eta) \,|\, \eta \in C(\bar{x})\} \tag{1.14}$$

Take δ and ϵ for $\bar{x}$ as in Lemma 1.42. Consider $K = \overline{B_1(\bar{x})}$, and S_K obtained in Lemma 1.21, one can choose a $\varrho \in (0,1)$ such that for $x \in B_\varrho(\bar{x})$,

$$|S_K(x,\cdot) - S_K(\bar{x},\cdot)|_{C^0} \le \epsilon/4.$$

Now let $y \in \mathbb{R}^d$ and $\lambda < 0$ be small such that $x_\lambda := \bar{x} + \lambda y \in B_\varrho(\bar{x})$ and $\lambda^2 < \epsilon/4$. Then by Lemma 1.42, for each x_λ, there is a descending cycle σ_λ such that

$$\max_{\sigma_\lambda} S(x_\lambda,\eta) \le R_S(x_\lambda) + \lambda^2$$

then

$$\max_{\sigma_\lambda} S(\bar{x},\eta) \le \max_{\sigma_\lambda} S(x_\lambda,\eta) + \frac{\epsilon}{4} \le R_S(x_\lambda) + \frac{\epsilon}{2} \le R_S(\bar{x}) + \frac{3\epsilon}{4}$$

and

$$R_S(\bar{x}) \le \max_{\sigma_\lambda \cap C_\delta(\bar{x})} S(\bar{x},\eta) = S(\bar{x},\eta_\lambda), \quad \text{for some } \eta_\lambda \in \sigma_\lambda \cap C_\delta(\bar{x}).$$

Hence we have

$$\begin{aligned} \lambda^{-1}[R_S(x_\lambda) - R_S(\bar{x})] &\le \lambda^{-1}[S(x_\lambda,\eta_\lambda) - S(\bar{x},\eta_\lambda)] - \lambda & (1.15)\\ &= \langle \partial_x S(x'_\lambda,\eta_\lambda), y\rangle - \lambda, & (1.16) \end{aligned}$$

where the last equality is given by the mean value theorem for some x'_λ in the line segment between $\bar{x}$ and x_λ.

Take the lim sup of both sides in the above inequality and let $\delta \to 0$, we get

$$\langle dR_S(\bar{x}), y\rangle \le \max_{\eta \in C(\bar{x})} \langle \partial_x S(\bar{x},\eta), y\rangle, \quad \forall y \in \mathbb{R}^d$$

Note that this implies that $dR_S(\bar{x})$ belongs to the sub-derivative of the convex function $f(y) := \max_{\eta\in C(\bar{x})}\langle \partial_x S(\bar{x},\eta), y\rangle$ at $v = 0$, [10] for which one can easily calculate

$$\partial f(0) = \mathrm{co}\{\partial_x S(\bar{x},\eta) : \eta \in C(\bar{x})\}.$$

Thus we get (1.14). In general,

$$\begin{aligned} \partial R_S(x) &= \mathrm{co}\{\lim_{x'\to x} dR_S(x')\} \subset \mathrm{co}\{\mathrm{co}\lim_{x'\to x}\{\partial_x S(x',\eta'), \eta' \in C(x')\}\} \\ &\subset \mathrm{co}\{\partial_x S(x,\eta), \eta \in C(x)\} \end{aligned}$$

by the upper-semi continuity of $x \mapsto C(x)$ and the continuity of $\partial_x S$. □

The formula (1.13) gives us somehow a *generalized graph selector*, while for a classical graph selector, we require that for almost every x,

$$dR_S(x) = \partial_x S(x,\eta), \quad \text{for some } \eta \in C(x)$$

from which $(x, dR_S(x)) \in L$.

Example 1.45. If S is a GFQI of $L = \varphi(dv) \in \mathcal{L}$ for $v \in C^2$, then $S_x := S(x,\cdot)$ is an excellent Morse function for almost every x, in which cases $C(x)$ consists of a single point, hence $\partial R_S(x) = \partial_x S(x,\eta)$ for a unique η, proving that R_S is a true graph selector for L.

Question 1.46. Is the minmax R_S also a true graph selector for $L \in \tilde{\mathcal{L}}$.

Question 1.47. Is it true that, when R_S is differentiable at x, one has $\big(x, dR_S(x)\big) \in L$? Here $L \in \mathcal{L}$ or even $\tilde{\mathcal{L}}$.

10. Recall that, for a convex function f, the sub-derivative at a point x is the set of ξ such that $f(y) - f(x) \ge \langle \xi, y - x\rangle, \forall y$

Chapter 2

Viscosity solutions and minmax solutions of Hamilton-Jacobi equations

We consider the Cauchy problem for the Hamilton-Jacobi equation:

$$\begin{cases} \partial_t u + H(t,x,\partial_x u) = 0 & \text{for } t \in (0,T] \\ u(0,x) = v(x) & \quad x \in \mathbb{R}^d \end{cases} \tag{H-J}$$

where $H \in C^2([0,T] \times T^*\mathbb{R}^d)$ and $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$ satisfy the condition of finite propagation speed. Unless otherwise specified, we assume that H has compact support (as a function on $[0,T] \times T^*\mathbb{T}^d$ when H and v are periodic).

2.1 Geometric solution and its minmax selector

From the geometric point of view of Lie and other mathematicians of the nineteenth century, a time-dependent first order partial differential equation $F(t,x,z,\frac{\partial z}{\partial t},\frac{\partial z}{\partial x}) = 0$ in d space variables is the hypersurface $E := \{F(t,x,z,e,p) = 0\}$ in the jet bundle $J^1(\mathbb{R} \times \mathbb{R}^d)$ endowed with the standard contact structure $\alpha := dz - edt - pdx = 0$. A C^1 function $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, is a solution if and only if its 1-jet $j^1u = \{(t,x,u(t,x),\partial_t u(t,x),\partial_x u(t,x))\}$ is contained in $\{F = 0\}$; note that $j^1 u$ is a Legendrian submanifold of $J^1(\mathbb{R} \times \mathbb{R}^d)$, i.e. a $d+1$-dimensional integral submanifold of the contact structure. In general, it is not possible to find such a global solution; the classical theory introduces generalized solutions called *geometric solutions*, which are the Legendrian submanifolds contained in E; at regular points, they are the $d+1$-dimensional integral submanifolds of the hyperplane field on E given by the intersection of the tangent plane and the contact plane. For the general theory, see [5, 4].

Being interested equations that do not depend on the values of the unknown function, we work in the cotangent bundle $T^*(\mathbb{R} \times \mathbb{R}^d)$ instead of the jet bundle. More precisely, under the hypotheses of (H-J), let

$$\mathcal{H}(t,x,e,p) =: e + H(t,x,p), \quad (t,x,e,p) \in T^*(\mathbb{R} \times \mathbb{R}^d)$$

and at the moment suppose that the initial function v is C^2.

Definition 2.1. Let $\varphi^s_{\mathcal{H}}$ denote the Hamiltonian flow of $\mathcal{H}$, which preserves the levels of $\mathcal{H}$, and let

$$\Gamma_v = \big\{(0, x, -H(0, x, dv(x)), dv(x))\big\} ;$$

then, the *geometric solution* of the Cauchy problem (H-J) is

$$L_{\mathcal{H},v} := \bigcup_{s\in[0,T]} \varphi^s_{\mathcal{H}}(\Gamma_v).$$

It is a Lagrangian submanifold containing the initial isotropic submanifold Γ_v and contained in the hypersurface

$$\mathcal{H}^{-1}(0) = \{(t, x, e, p)|e + H(t, x, p) = 0\} \subset T^*(\mathbb{R} \times \mathbb{R}^d).$$

As every Lagrangian submanifold L of $T^*(\mathbb{R}\times\mathbb{R}^d)$ contained in $\mathcal{H}^{-1}(0)$ is locally invariant by $\varphi^s_{\mathcal{H}}$, this geometric solution is in some sense maximal.

Writing $T^*(\mathbb{R} \times \mathbb{R}^d)$ as $T^*\mathbb{R} \times T^*\mathbb{R}^d$, we have $X_{\mathcal{H}} = (1, -\partial_t H, X_H)$, and

$$L_{\mathcal{H},v} = \Big\{\Big(t, -H(t, \varphi^t_H(dv)), \varphi^t_H(dv)\Big) , t \in [0, T]\Big\}$$

where $\varphi^t_H := \varphi^{0,t}_H$ is the Hamiltonian isotopy generated by H.

Lemma 2.2. *Formula* (1.5) *defines a GFQI*

$$S : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{2l} \to \mathbb{R}$$

of $L_{\mathcal{H},v}$.

Proof. For simplicity, we may assume that $T \in (0, \delta_H)$, hence that

$$S : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{2d} \to \mathbb{R}, \quad S(t, x, x_0, y_0) = v(x_0) + xy_0 + \phi^t_H(x, y_0) - x_0y_0.$$

Let $(x_0, y_0) \in \Sigma_S$, then

$$(\partial_t S(t, x, x_0, y_0), \partial_x S(t, x, x_0, y_0)) = (\partial_t \phi^t_H(x, y_0), \partial_x \phi^t_H(x, y_0)) = (-H(t, x, y(t)), y(t)),$$

where $(x, y(t)) = \varphi^t_H(x_0, y_0)$ with $y_0 = dv(x_0)$.

Hence

$$\{(t, x, \partial_t S(t, x, x_0, y_0), \partial_x S(t, x, x_0, y_0))|(x_0, y_0) \in \Sigma_S\} = L_{\mathcal{H},v}$$

□

If there exists a C^1 function $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that

$$L = L_{\mathcal{H},v} = \{(t, x, \partial_x u(t, x), \partial_x u(t, x))\} \subset T^*([0, T] \times \mathbb{R}^d)$$

we say that L is a 1-graph in $T^*([0, T] \times \mathbb{R}^d)$. In this case, u is a global solution of the Cauchy problem of (H-J) equation. In general, L may be the graph of the derivatives of a multi-valued function.

Definition 2.3. A *caustic point* for the geometric solution L is a point $(t, x) \in [0, T] \times \mathbb{R}^d$ at which the projection $\pi : L \to [0, T] \times \mathbb{R}^d$, $(x, p) \mapsto x$ is singular.

The caustic points are the obstacles preventing L from being (locally) a 1-graph. Indeed, if (t,x) is a regular value of the map $\pi : L \to [0,T]\times\mathbb{R}^d$, then by the inverse mapping theorem, there is a diffeomorphism f from a neighborhood of (t,x) to a neighbourhood of a in L for each $a \in \pi^{-1}(t,x)$; thus, near a, the submanifold L is the graph of the function f (composed with the inclusion $L \hookrightarrow T^*([0,T]\times\mathbb{R}^d)$) and therefore, being Lagrangian, the graph of the derivative of a function—which is a local solution of the equation.

Definition 2.4. The *big wave front* of the geometric solution L is defined as

$$\tilde{\mathcal{F}} := \left\{ \big(t,x,S(t,x;\eta)\big) \mid \tfrac{\partial S}{\partial \eta}(t,x;\eta) = 0 \right\} \subset J^0(\mathbb{R}\times\mathbb{R}^d)$$

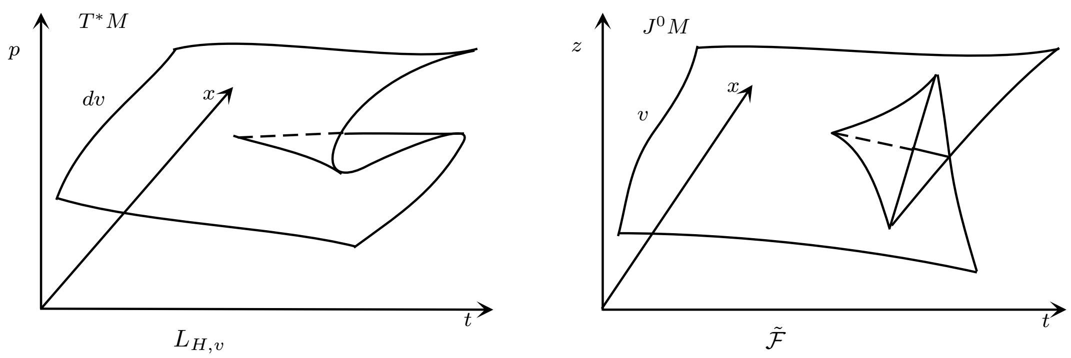
and we call the restriction of $\tilde{\mathcal{F}}$ at each time t a *wave front*, denoted by $\mathcal{F} \subset J^0(\mathbb{R}^d)$.

The big wave front $\tilde{\mathcal{F}}$ is independent of the choice of the GFQI of L, up to a vertical translation. Indeed L, as an exact Lagrangian submanifold, can be lifted to a Legendrian submanifold of $J^1(\mathbb{R}\times\mathbb{R}^d)$, unique up to vertical translation, and the big wave front is the projection of this Legendrian submanifold from $J^1(\mathbb{R}\times\mathbb{R}^d)$ to $J^0(\mathbb{R}\times\mathbb{R}^d)$.

An equivalent but more economic way to describe the geometric solution is to identify each $\varphi^s_{\mathcal{H}}(\Gamma_v)$ with $\{s\}\times\varphi^s_H(dv)$ by the inverse of the map $(t,x,p)\mapsto(t,x,-H(t,x,p),p)$. In this way, we also call the union

$$L_{H,v} := \bigcup_{t\in[0,T]} \{t\}\times\varphi^t_H(dv) \subset \mathbb{R}\times T^*\mathbb{R}^d$$

a geometric solution.



If we look at the projection of the characteristics, that is, the image of the graph of the solutions $\{(t,\varphi^t_H(x_0,p_0))\}_{t\in[0,T]}$, $(x_0,p_0)\in T^*\mathbb{R}^d$, of Hamilton's equations under the projection

$$\pi : [0,T]\times T^*\mathbb{R}^d \to \mathbb{R}\times\mathbb{R}^d, \quad (t,x,p)\mapsto(t,x).$$

then L is not a 1-graph when the corresponding characteristics intersect under the projection. Without ambiguity, we will simply say that the characteristics intersect.

Now, as before, we consider more generally the Lipschitz case. Given $v\in C^{\mathrm{Lip}}(\mathbb{R}^d)$, set

$$\Gamma_v = \{(0,x,-H(0,x,p),p) : p\in\partial v(x)\}$$

and similarly

$$\begin{aligned} L_{\mathcal{H},v} &= \bigcup_{s\in[0,T]} \varphi^s_{\mathcal{H}}(\Gamma_v) = \Big\{ \Big(t, -H\big(t,\varphi^t_H(x,p)\big), \varphi^t_H(x,p)\Big) : p\in\partial v(x),\ t\in[0,T] \Big\} \\ L_{H,v} &= \bigcup_{t\in[0,T]} \{t\}\times\varphi^t_H(\partial v) := \bigcup_{t\in[0,T]} \{t\}\times\{\varphi^t_H(x,p) : p\in\partial v(x)\} \end{aligned}$$

where ∂ is Clarke's generalized derivative. We call $L_{\mathcal{H},v}$ or $L_{H,v}$ *generalized geometric solutions*. They are also generated by the GFQI given by formula (1.5) on page 18. Similarly, we can define the generalized wave fronts and big wave front.

Definition 2.5. For any time $0 \le s < t \le T$, we define the minimax operator [1]

$$R_H^{s,\tau} : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d), \quad \tau \in [s,t]$$

for the (H-J) equation as

$$R_H^{s,\tau} v(x) = \inf \max_{\eta} S(\tau, x, \eta)$$

where $S : [s,t] \times \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}$ is given by (1.5).

For completeness, without referring to the uniqueness theorem for GFQI's, we give a proof that the minmax is well-defined independently of the subdivisions.

Lemma 2.6. *The minmax $R_S(x) = \inf \max S(x,\eta)$ given by* (1.4) *or* (1.5) *is independent of the subdivision of time in the construction of S.*

Proof. First assume $t - s < \delta_H$; given $\tau \in (s,t)$, consider the family of subdivisions $\zeta_\mu := \{s \le s + \mu(\tau - s) < t\}$; then,

$$S_\mu(x; x_0, y_0, x_1, y_1) = v(x_0) + \phi_H^{s,s+\mu(\tau-s)}(x_1, y_0) + (x_1 - x_0)y_0 + \phi_H^{s+\mu(\tau-s),t}(x, y_1) + (x_2 - x_1)y_1\,,$$

where $x_2 := x$, is the generating family defined by (1.4) and associated to ζ_μ, $\mu \in (0,1]$. The function S_μ is continuous in μ and the minmax $R_{S_\mu}(x)$ is the critical value of the map $\eta \mapsto S_\mu(x;\eta)$ with $\eta := (x_0, y_0, x_1, y_1)$. By 1.6, the set of all such critical values is independent of μ; as it has measure zero by Sard's Theorem, R_{S_μ} is constant for $\mu \in [0,1]$. In particular, letting $x_1' := x_1 - x_0$ and $y_0' = y_0 - y_1$, we get

$$S_0(x; x_0, y_0, x_1, y_1) = S_0(x; (x_0, y_1, x_1', y_0')) = v(x_0) + \phi_H^{s,t}(x_2, y_1) + (x_2 - x_0)y_1 + x_1' y_0'.$$

It is obtained by adding the quadratic form $x_1' y_0'$ to

$$S(x; x_0, y_1) = v(x_0) + \phi_H^{s,t}(x_2, y_1) + (x_2 - x_0)y_1,$$

which is the generating family related to ζ_0. We conclude that $R_S(x) = R_{S_0}(x) = R_{S_1}(x)$.

In general, given any two subdivisions ζ', ζ'' of $[s,t]$ with [2] $|\zeta'|, |\zeta'| < \delta_H$, denote by $\zeta = \zeta' \cup \zeta'' = \{s = t_0 < \cdots < t_n = t\}$ the subdivision obtained by collecting the points in ζ' and ζ''. If t_j is not contained in ζ', we consider the family of subdivisions

$$\zeta_\mu(j) = \{t_0 < t_{j-1} \le t_{j-1} + \mu(t_j - t_{j-1}) < t_{j+1} < \cdots t_n\}, \quad \mu \in [0,1]$$

The same argument as before shows that the minmax relative to $\zeta_0(j)$ and $\zeta_1(j)$ are the same. Continuing this procedure, we get that the minmax relative to ζ' and ζ are the same, and the same holds for ζ'' and ζ. Therefore the minmax with respect to ζ' and ζ'' are the same. □

Proposition 2.7 ([18])**.** *If $v \in C^2 \cap C^{\mathrm{Lip}}(\mathbb{R}^d)$, then $R_H^{0,t} v(x)$ verifies the (H-J) equation almost everywhere.*

---

1. The inclusion $R_H^{s,\tau}\big(C^{\mathrm{Lip}}(\mathbb{R}^d)\big) \subset C^{\mathrm{Lip}}(\mathbb{R}^d)$ is proven in Proposition 2.47 p. 44.
2. For a subdivision $\zeta = \{t_0 < \cdots < t_n\}$, we let $|\zeta| := \max_i |t_{i+1} - t_i|$.

Proof. This is a direct consequence of the fact that S is a GFQI of $L_{\mathcal{H},v}$ and the minmax is a graph selector in this case. $\square$

In general, for a Lipschitzian initial function, we do not know whether the minmax verifies the equation almost everywhere or not. But in view of the estimation of generalized derivatives in Proposition 1.44, we still call $R_H^{s,t}v(x)$ the (generalized) *minmax solution* of the Cauchy problem of (H-J) equation.

Lemma 2.8. *If v is C^2 with bounded second derivative, then there exists a $\epsilon > 0$ such that for $t \in [0, \epsilon)$, the minmax $R_H^{0,t}v(x)$ is C^2.*

Proof. We will show that, there exists a $\epsilon > 0$, such that for $t \in (0, \epsilon)$, the characteristics beginning from the graph dv do not intersect. More precisely, with the notation introduced in Subsection 1.2.1, the map $f_t : x_0 \mapsto X_0^t(x_0, dv(x_0))$ is a diffeomorphism. Indeed, for t small enough,

$$\mathrm{Lip}(f_t - Id) \leq \mathrm{Lip}(\alpha_0^t - Id)(1 + \mathrm{Lip}\, dv) \leq (e^{c_H t} - 1)(1 + \mathrm{Lip}\, dv) < 1$$

where α_0^t and c_H are defined in Lemma 1.12. This in turn means that the projection map $L = \varphi_H^t(dv) \to \mathbb{R}^d$, $(x, p) \mapsto x$ is a diffeomorphism, hence $L = \{x, dR_H^{0,t}v(x)\}$, from which we obtain that $R_H^{0,t}v(x)$ is C^2. $\square$

2.2 Viscosity solution of (H-J) equation

As we have seen in the previous section, there are in general no global classical C^1 solutions of the Cauchy problem (H-J), due to the crossing of characteristics. The only solutions that exist are "weak solutions" in the sense of distributions, for example functions verifying the equation almost everywhere. However, such solutions are not unique. Different attempts have been made, adding conditions on weak solutions to ensure uniqueness and, of course, some physical meaning. Roughly, there are two directions in which the pioneers worked: for conservation laws, there are entropy conditions, such as Oleinik's in dimension one [48], and Kružkov's for general dimensions [47]; for equations with convex Hamiltonian (or initial functions), there are the explicit solution constructed by Hopf formula [41] for conservation laws and the Lax-Oleinik formula for general Hamiltonians, which are widely used in weak KAM theory, see for example [33].

In the 1980's, M. G. Crandall, L. C. Evans, and P. L. Lions introduced the notion of "viscosity solution" for general nonlinear first order partial differential equations [50, 28]. Viscosity solutions need not be differentiable anywhere, which makes their relationship with the classical crossing of characteristics unclear. However, they possess very general existence, uniqueness and stability properties and, in a large class of "good" cases, they coincide with the weak solutions introduced before them.

Definition 2.9. A function $u \in C^0\big((0, T) \times \mathbb{R}^d\big)$ is called a *viscosity subsolution* (resp. *supersolution*) of

$$\partial_t u + H(t, x, \partial_x u) = 0$$

when it has the following property: for every $\psi \in C^1\big((0, T) \times \mathbb{R}^d\big)$ and every point (t, x) at which $u - \psi$ attains a local maximum (resp. minimum), one has

$$\partial_t \psi + H(t, x, \partial_x \psi) \leq 0, \quad (\text{resp. } \geq 0).$$

The function u is a *viscosity solution* if it is both a viscosity subsolution and supersolution.

Remarks As their derivatives do not appear in the definition, the test functions ψ can be assumed to satisfy $\psi \geq u$ (resp. $\psi \leq u$) near (t,x), with equality at (t,x).

One can replace C^1 test functions ψ by C^∞ test functions in the definition: indeed, if one does so and $u-\psi$ has, e.g., a maximum at (t_0,x_0) for some C^1 function ψ then, by adding to ψ a nonnegative smooth function vanishing only at (t_0,x_0), one can assume that the maximum is strict; if we approximate ψ by C^∞ functions ψ_n in the C^1 topology then, in a fixed compact neighbourhood U of (t_0,x_0), the function $u-\psi_n$ reaches its maximum at a point (t_n,y_n) interior to U for large enough n, hence $\partial_t\psi_n(t_n,x_n)+H(t_n,y_n,d\psi_n(y_n)) \leq 0$; as $(t_n,y_n) \to (t_0,x_0)$ when $n \to \infty$, this does yield $\partial_t\psi(t_0,x_0)+H(t_0,x_0,d\psi(x_0)) \leq 0$. Obviously, a classical C^1 solution is a viscosity solution. Indeed, we have

Lemma 2.10. *A viscosity solution verifies the equation wherever it is differentiable.*

Proof. Just observe that, if $\psi - u$ reaches a local minimum (resp. maximum) at a point (t,x) where u is differentiable, then $d\psi(t,x) = du(t,x)$ and use the definition. ☐

A more intrinsic way to define viscosity solutions is to introduce the notion of lower and upper differentials. Let M denote a general manifold.

Definition 2.11. Let $u : M \to \mathbb{R}$ be a function; for each $x_0 \in M$, the set of lower differentials of u at $x_0 \in M$ is (in any chart)

$$D^-u(x_0) := \left\{ p \in T^*_{x_0}M : \lim \inf_{x\to x_0} \frac{u(x)-u(x_0)-p(x-x_0)}{\|x-x_0\|} \geq 0 \right\}$$

Similarly, the set of upper differentials of u at x_0 is

$$D^+u(x_0) := \left\{ p \in T^*_{x_0}M : \lim \sup_{x\to x_0} \frac{u(x)-u(x_0)-p(x-x_0)}{\|x-x_0\|} \leq 0 \right\}.$$

For example, if $M = \mathbb{R}$ and $u(x) = |x|$, then $D^-u(x_0) = [-1,1]$ and $D^+u(x_0) = \emptyset$.

A reference for the following results is [8].

Lemma 2.12 ([8]). *i) If $\psi : M \to \mathbb{R}$ is differentiable at x and such that $\psi \leq u$ (resp. $\psi \geq u$) with equality at x, then $d\psi(x) \in D^-u(x)$ (resp. $D^+u(x)$).*

ii) For each $p \in D^-u(x)$ (resp. $D^+u(x)$), there exists $\psi \in C^1(M,\mathbb{R})$ such that $d\psi(x) = p$ and $\psi \leq u$ (resp. $\psi \geq u$) in a neighborhood of x, with equality at x.

Remark 2.13. In (ii), it is not always possible to find a function of class C^k with $k > 1$. A counterexample can be given as follows: let $u : \mathbb{R} \to \mathbb{R}$, $u = |x|^\alpha$ with $\alpha \in (1,2)$, if ψ_+ is a function such that $\psi_+ \geq u$ and $\psi_+(0) = u(0) = 0$, then we have $\psi_+'(0) = 0$, and

$$\psi_+''(0) = \lim_{x\to 0} 2\frac{\psi_+(x)}{x^2} \geq \lim_{x\to 0} |x|^{\alpha-2} = +\infty.$$

That is, ψ_+ can not be C^2 at 0.

Lemma 2.14 ([8]). *i) Both $D^+u(x)$ and $D^-u(x)$ are closed convex subsets of T^*_xM;*

ii) The subsets $D^\pm u(x)$ are both non-empty if and only if u is differentiable at x, in which case

$$D^+u(x) = D^-u(x) = \{du(x)\}.$$

iii) One has $D^+u(x) \cup D^-u(x) \subseteq \partial u(x)$.

Remark 2.15. When u is convex, $D^-u(x) = \partial u(x)$ is the sub-derivative of u in classical convex analysis.

The following result is a corollary of Lemma 2.12:

Proposition 2.16. *A function $u \in C^0\big((0,T) \times \mathbb{R}^d\big)$ is a viscosity subsolution (resp. supersolution) of the (H-J) equation if and only if, for all $(t,x) \in (0,T) \times \mathbb{R}^d$ and $(e,p) \in D^+u(t,x)$ (resp. $(e,p) \in D^-u(t,x)$),*

$$e + H(t,x,p) \leq 0 \quad (\textit{resp.}\ \geq 0).$$

Remark 2.17. It follows that the notion of a viscosity solution is local.

The existence of viscosity solutions is ensured by the so-called "vanishing viscosity method" at the origin of the name "viscosity". The approach is to consider the approximate problem

$$(\mathrm{HJ}_\epsilon) \qquad \begin{cases} \partial_t u^\epsilon + H(t,x,\partial_x u^\epsilon) = \epsilon \Delta u^\epsilon \\ u^\epsilon(0,x) = v(x) \end{cases}$$

for $\epsilon > 0$. This quasilinear parabolic Cauchy problem turns out to have a smooth solution u^ϵ, as the viscosity term $\epsilon\Delta$ regularizes the Hamilton-Jacobi equation. In practice, the family $\{u^\epsilon\}_{\epsilon>0}$ is uniformly bounded and equicontinuous on compact subsets of $\mathbb{R} \times \mathbb{R}^d$. Consequently, by the Arzela-Ascoli Theorem, every sequence ϵ_n of positive numbers converging to 0 has a subsequence ϵ_{n_k} such that $u^{\epsilon_{n_k}}$ converges to a limit function u.

Proposition 2.18 ([28])**.** *Such a limit u is a viscosity solution of the* (H-J) *problem.*

Uniqueness follows at once from the following estimate:

Proposition 2.19 ([29])**.** *If u_1 and u_2 are viscosity solutions of the Hamilton-Jacobi equation, then*

$$\sup_{x\in\mathbb{R}^d} \big(u_1(t,x) - u_2(t,x)\big)^+ \leq \sup_{x\in\mathbb{R}^d} \big(u_1(0,x) - u_2(0,x)\big)^+.$$

This proposition gives more than uniqueness: it provides a monotonicity property for viscosity solutions with respect to the initial condition: if $u_1(0,\cdot) \leq u_2(0,\cdot)$, then $u_1 \leq u_2$.

Theorem 2.20 (Stability,[28])**.** *Suppose that the sequences of functions $H^n : \mathbb{R} \times T^*\mathbb{R}^d \to \mathbb{R}$ and $u^n : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ converge uniformly on compact subsets to H and u respectively. If each u^n is a viscosity solution of*

$$\partial_t u^n + H^n(t,x,\partial_x u^n) = 0,$$

then u is a viscosity solution of

$$\partial_t u + H(t,x,\partial_x u) = 0.$$

Theorem 2.21 ([29])**.** *If $v \in C^{\mathrm{Lip}}(\mathbb{R}^d))$ and $H \in C^2_c([0,T] \times T^*\mathbb{R}^d)$, then there exists a unique viscosity solution of the Cauchy problem of the Hamilton-Jacobi equation. Moreover, this solution is globally Lipschitz.*

A notable feature of the viscosity solution, is that it is *Markovian*, meaning that, if

$$J_s^t : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d)$$

denotes the viscosity solution operator (for a fixed Hamiltonian) which to v associates the time t of the solution equal to v at time s, then the "two-parameter groupoid" property

$$J_\tau^t = J_s^t \circ J_\tau^s$$

is satisfied. This follows easily from uniqueness.

2.3 The convex case: Lax-Oleinik semi-groups and viscosity solutions

As mentioned before, the viscosity solutions of Hamilton-Jacobi equations whose Hamiltonian is convex in p have been well studied. This section is a brief survey of the convex theory in terms of generating families, a finite dimensional version of the action functional. See [20, 45, 12, 9, 33]. We consider the following model case:

1. $H(t,x,p) \in C^2([0,T]\times T^*\mathbb{R}^d)$ is strictly convex in p; more precisely, $\partial^2_{pp}H$ is uniformly positively definite, meaning that $\partial^2_{pp}H(t,x,p)\xi^2 \geq c|\xi|^2$ for some positive constant c.
2. $H_t(x,p) = \frac{1}{2}|p|^2$ off a compact subset.
3. The initial datum v is Lipschitzian.

Remark 2.22.
- The functions H and v verify the condition of finite propagation speed. Indeed, it is easy to see that they satisfy the conditions in Proposition 1.32.
- If the base manifold is $\mathbb{T}^d$, and H is a Tonelli Hamiltonian (i.e., satisfies 1.) then, for any Lipschitz initial function v, $\mathcal{U} := \bigcup_{t\in[0,T]}\{t\} \times \varphi_H^{0,t}(\partial v)$ is compact and we can modify H without changing $\mathcal{U}$ so that 2. is satisfied.

With the notation of subsection 1.2.1 p. 15, the following construction of generating families is taken from [20], section 2.3.

Lemma 2.23 ([20]). *There exists a constant $\epsilon_H > 0$ such that for any $0 < |s-t| < \epsilon_H$, the map*

$$\beta_s^t : (x,y) \to (x, X_s^t)$$

is a diffeomorphism.

Definition 2.24. A diffeomorphism $\varphi : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ admits a *classical generating function* ψ, if $\psi : (\mathbb{R}^d)^2 \to \mathbb{R}$ is C^1, such that $((x,y),(X,Y)) \in \mathrm{Graph}(\varphi)$ if and only if

$$\begin{cases} Y = \partial_X \psi(X,x) \\ y = -\partial_x \psi(X,x). \end{cases}$$

Lemma 2.25. *For $0 < |t-s| < \epsilon_H$, the transformation $\varphi_s^t = (X_s^t, Y_s^t)$ of H admits the classical generating function*

$$\psi_s^t(X_s^t, x) = \int_s^t \Big(Y_s^\tau \dot{X}_s^\tau - H(\tau, X_s^\tau, Y_s^\tau)\Big)d\tau,$$

which satisfies $\psi_s^t(X,x) = \frac{1}{2(t-s)}|X-x|^2$ for large enough $|X-x|$.

Proof. As proved in Lemma 1.14,

$$d\psi = Y_s^t dX_s^t - ydx,$$

so ψ_s^t is a classical generating function by Lemma 2.23. Moreover, since $H_t(x,y) = \frac{1}{2}|y|^2$ outside a compact subset K, as

$$X - x = X_s^t - x = \int_s^t \partial_y H(\tau, X_s^\tau, Y_s^\tau)d\tau,$$

for $|X-x|$ large enough, we must have $H(s,x,y) = |y|^2/2$, hence $\varphi_s^t(x,y) = (x+(t-s)y^\sharp, y)$ and $\psi_s^t(X,x) = \frac{1}{2(t-s)}|X-x|^2$. □

Remark 2.26. The relation between the generating functions ϕ_s^t and ψ_s^t is that:

$$\phi_s^t(X,y) = \psi_s^t(X,x) - y(X-x)$$

where (X,x) is obtained from (X,y) via the diffeomorphism $\beta_s^t \circ \alpha_t^s$:

$$(X,y) \xrightarrow{\alpha_t^s} (x,y) \xrightarrow{\beta_s^t} (x,X).$$

The following result, essentially due to Hamilton, shows that our convexity asumption on H divides by two the number of fiber variables needed in generating families:

Lemma 2.27 (Composition formula 3,[20])**.** *If ψ_1 and ψ_2 are classical generating functions for two diffeomorphisms φ_1, $\varphi_2 : T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ respectively, then $\varphi_2 \circ \varphi_1$ admits the classical generating family*

$$\psi(x_2,x_0;x_1) = \psi_1(x_1,x_0) + \psi_2(x_2,x_1)$$

in the sense that $((x_0,y_0),(x_2,y_2)) \in Graph(\varphi_2 \circ \varphi_1)$ iff there exists x_1 such that

$$\begin{cases} y_2 = \partial_{x_2}\psi(x_2,x_0;x_1) \\ y_0 = -\partial_{x_0}\psi(x_2,x_0;x_1) \\ 0 = \partial_{x_1}\psi(x_2,x_0;x_1) \end{cases}$$

Lemma 2.28 ([45])**.** *For any $0 < s < t \leq T$, the subset $L = \varphi_s^t(\partial v)$ has the generating family given by*

$$F_s^t(x;(x_i)_{0\leq i\leq j}) = v(x_0) + \Psi_s^t(x,(x_i)) := v(x_0) + \sum_{0\leq i\leq j} \psi_{\tau_i}^{\tau_{i+1}}(x_{i+1},x_i) \tag{2.1}$$

where $x_{j+1} := x$ and $\{s = \tau_0 < \tau_1 < \cdots < \tau_{j+1} = t\}$ is a subdivision of $[s,t]$ such that $|\tau_i - \tau_{i+1}| < \epsilon_H$, $0 \leq i \leq j$. Up to diffeomorphism, F_s^t is quadratic of index 0 at infinity.

Proof. As in the composition formula, it is easy to verify that F_s^t is a generating family. We claim that it is equivalent to a generating family quadratic at infinity with a quadratic form of Morse index zero. By the change of variables $\xi_i = x_{i+1} - x_i$, $0 \leq i \leq j$, we get the generating family

$$\tilde{F}_s^t(x;(\xi_i)_{0\leq i\leq j}) := v(x - \sum_{0\leq k\leq j} \xi_k) + \sum_{0\leq i\leq j} \psi_{\tau_i}^{\tau_{i+1}}(x - \sum_{i+1\leq k\leq j} \xi_k, x - \sum_{i\leq k\leq j} \xi_k)$$

and, for $|\xi_i|$, $0 \leq i \leq j$ large enough, we have

$$\tilde{F}_s^t(x,(\xi_i)_{0\leq i<j}) := v(x - \sum_{0\leq i\leq j} \xi_k) + \sum_{0\leq i\leq j} \frac{1}{2(\tau_{i+1}-\tau_i)}|\xi_i|^2$$

where the second term is a nondegenerate quadratic form of Morse index zero. $\square$

Notation. To simplify, we let $F := F_s^t$ the function defined by (2.1), and R_F denotes the minmax function.

Lemma 2.29. *The minmax is reduced to min, i.e.*

$$R_F(x) = \min_\eta F(x,\eta).$$

Proof. As Q is of Morse index zero, the descending cycles are points. □

Corollary 2.30. *For S and F defined by (1.5) and (2.1) respectively, we have*

$$\inf \max S(x,(x_i,y_i)) = \min F(x,(x_i)).$$

Proof. In view of Lemma 2.29, for the case where v is C^2, we can conclude by the uniqueness theorem of GFQI (ref. Theorem 1.4) since both S and F generates the same Lagrangian submanifold $L = \varphi(dv)$. In general, for v Lipschitz, we can apply the continuity dependence of the minmax selector on the generating family (ref. Lemma 1.29). □

Hence, we are ready to define:

Definition 2.31. The min operator for the (H-J) equation is defined by

$$R_s^t : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d), \quad R_s^t v(x) = \min_{(x_i)} F_s^t(x;(x_i)).$$

Note that R_s^t is defined independently of the choice of subdivisions of $[s,t]$.

Lemma 2.32. *If $\bar\eta$ is a point realizing the minimum $R_F(x) = \min_\eta F(x,\eta)$, then we have $\partial_x F(x,\bar\eta) \in D^+R_F(x)$. When R_F is differentiable at x, it follows that $dR_F(x) = \partial_x F(x,\bar\eta)$ and, moreover, $\bar\eta$ is unique.*

Proof. Since R_F is the minimum of F, we have $R_F(x') \leq F(x',\bar\eta)$ for $x' \neq x$, hence x is a local minimum point for $F(x',\bar\eta) - R_F(x')$, hence $\partial_x F(x,\bar\eta) \in D^+R_F(x)$ by Lemma 2.12. When R_F is differentiable, the uniqueness of $\bar\eta$ follows: indeed, if $\bar\eta = (\bar x_i)_{0\leq i\leq j}$ the relation $dR_F(x) = \partial_x F(x,\bar\eta)$ writes $y := dR_F(x) = \partial_1\psi_{\tau_j}^{\tau_{j+1}}(x,\bar x_j)$, which determines $\bar x_j = \pi_2 \circ \beta_{\tau_{j+1}}^{\tau_j}(x,y)$; if $j = 0$, this proves uniqueness; otherwise, as $F(x,\bar\eta)$ is minimal and differentiable with respect to x_j, we have $\partial_{x_j} F(x,\bar\eta) = 0$, i.e. $\bar y_j := -\partial_2\psi_{\tau_j}^{\tau_{j+1}}(x,\bar x_j) = \partial_1\psi_{\tau_{j-1}}^{\tau_j}(\bar x_j,\bar x_{j-1})$, which determines $\bar x_{j-1} = \pi_2 \circ \beta_{\tau_j}^{\tau_{j-1}}(\bar x_j,\bar y_j)$, etc. □

Lemma 2.33. *We have*

$$\partial R_F(x) = co\{\partial_x F(x,\eta) : \eta \in C(x)\} = D^+R_F(x)$$

where $C(x) = \{\eta : R_F(x) = F(x,\eta)\}$. Moreover, if $R_F(x)$ is differentiable at x, then it is C^1 at x with respect to the set of those points at which it is differentiable.

Proof. The last assertion is Remark A.2 in Appendix A. As $D^+R_F(x) \subset \partial R_F(x)$ and $D^+R_F(x)$ is a convex set, we have

$$\mathrm{co}\{\partial_x F(x,\eta) : \eta \in C(x)\} \subset D^+R_F(x) \subset \partial R_F(x).$$

For the reverse inclusion, we use the fact that $dR_F(x) = \partial_x F(x,\eta)$ when R_F is differentiable at x and $F(x,\eta) = R_F(x)$: as $\partial R_F(x)$ is the convex hull of the set of limits of convergent sequences $dR_F(x_n)$ with $\lim x_n = x$, it is the convex hull of the set of limits of convergent sequences $\partial_x F(x_n,\eta_n)$ with $\lim x_n = x$ and $F(x_n,\eta_n) = R_F(x_n)$; since $\partial_x F$ is continuous, this does yield $\partial R_F(x) \subset \mathrm{co}\{\partial_x F(x,\eta) : \eta \in C(x)\}$. □

As a consequence, we know that the min function $R_0^t v(x)$ does define a graph selector for the geometric solution $\varphi_0^t(\partial v)$, even for Lipschitz initial functions. Hence $R_0^t v(x)$ satisfies the (H-J) equation almost everywhere. In this sense, we will call it a min solution, or min solution operator.

Example 2.34. Let $H = H(p)$, with $\partial^2_{pp}H \geq cI$ for some $c > 0$. Its flow is $\varphi^t(x, y) = (x + tH'(y), y)$, hence has a classical generating function

$$\begin{aligned} \psi^t(X, x) &= t(yH'(y) - H(y)), \quad \text{with}\, H'(y) = \frac{X - x}{t} \\ &= t \max_p \,(p\frac{X - x}{t} - H(p)) = tH^*(\frac{X - x}{t}) \end{aligned}$$

where H^* is the Legendre tranformation of H. Then we get the min function

$$R^t_0 v(x) = \min_{x_0} \,(v(x_0) - tH^*(\frac{x - x_0}{t})), \quad t > 0$$

which is the Hopf formula.

Proposition 2.35. *The min solution operator is a semigroup*[3] *with respect to time, that is,*

$$R^t_0 v(x) = R^t_s \circ R^s_0 v(x), \quad 0 \leq s \leq t$$

Proof.

$$\begin{aligned} R^t_s \circ R^s_0 v(x) &= \min_{(x_i)} \Big(R^s_0 v(x_0) + \Psi^t_s(x, (x_i))\Big) \\ &= \min_{(x_i)} \Big(\min_{(x'_j)}(v(x'_0) + \Psi^s_0(x_0, (x'_j))) + \Psi^t_s(x, (x_i))\Big) \\ &= \min_{(x_i),(x'_j)} \Big(v(x'_0) + \Psi^t_0(x, (x_i), (x'_j))\Big) = R^t_0 v(x) \end{aligned}$$

where the last equality is due to the fact that the min is independent of the subdivision. □

The semi-group property in the convex case has the following stronger geometric interpretation:

Proposition 2.36. *For $t > 0$, and any $s \in (0, t)$, we have*

$$\overline{dR^t_0 v} \subset \varphi^t_s(dR^s_0 v).$$

Proof. A priori, by Lemma 2.32 we have $dR^t_0 v \subset \varphi^t_s(\partial R^s_0 v)$. By the semi-group property of both the operator R and the flow φ, it is enough to consider $0 < t - s < \epsilon_H$. Fixing x, suppose x_1 is a minimizing point for

$$R^t_0 v(x) = \min_{x'_1} \,(R^s_0 v(x'_1) + \psi^t_s(x, x'_1))$$

and x_0 a minimizing point for

$$R^s_0 v(x_1) = \min_{x'_0} \,(v(x'_0) + \psi^s_0(x_1, x'_0)).$$

Take

$$f_+(y) := v(x_0) + \psi^s_0(y, x_0), \quad f_-(y) := R^t_0 v(x) - \psi^t_s(x, y)$$

then $f_\pm$ is C^1 and $f_- \leq R^s_0 v \leq f_+$ with equality at $y = x_1$, hence $R^s_0 v$ is differentiable at x_1 and

$$dR^s_0 v(x_1) = df_\pm(x_1) = \partial_1 \psi^s_0(x_1, x_0) = -\partial_2 \psi^t_s(x, x_1).$$

3. Or rather a“two-parameter groupoid”.

By the definition of classical generating function, we have

$$(x, \partial_1 \psi_s^t(x, x_1)) = \varphi_s^t(x_1, -\partial_2 \psi_s^t(x, x_1)) = \varphi_s^t(x_1, dR_0^s(x_1)), \tag{2.2}$$

In particular, if $R_0^t v$ is differentiable at x,

$$(x, dR_0^t v(x)) = (x, \partial_1 \psi_s^t(x, x_1)) = \varphi_s^t(x_1, dR_0^s(x_1))$$

where the second equality follows from the definition of classical generating function.

Now let $(x, p) = \lim (x^n, dR_0^t v(x^n))$; if one sets $(x_1^n, p_1^n) := \varphi_t^s(x^n, dR_0^t v(x^n))$ and $(x_1, p_1) = \lim(x_1^n, p_1^n)$, then

$$R_0^t v(x^n) = R_0^s v(x_1^n) + \psi_s^t(x^n, x_1^n),$$

hence

$$R_0^t v(x) = R_0^s v(x_1) + \psi_s^t(x, x_1).$$

Now that x_1 is a minimizing point of $R_0^t v(x)$, applying the same argument as before, we get $(x, p) = \varphi_s^t(x_1, dR_0^s v(x_1))$. □

Corollary 2.37. *For any $t > 0$ and $s \in (0, t)$,*

$$M_t(v) := \{(x, p) | p = \partial_x F_0^t(x, \eta), F_0^t(x, \eta) = R_0^t v(x)\} \subset \varphi_s^t(dR_0^s v).$$

Proof. We first remark that $M_t(v)$ is independent of the subdivisions used to define $F_0^t(x, \eta)$. We have proved the claim for $|t - s| < \epsilon_H$ in Proposition 2.36, see (2.2). The general case where the generating family F_0^t has more fiber variables can be proved similarly. □

Remark 2.38.

1. For $s > 0$, the min solution operator can be defined for bounded uniformly continuous v and $R_0^s v(x)$ turns out to be Lipschitzian and semi-concave.

2. Proposition 2.36 tells us that the differential of the min solution at time t, a priori extracted from the geometric solution $\varphi_s^t(\partial v_s)$ (by Lemma 2.33), indeed lies in $\varphi_s^t(dv_s)$. Roughly speaking, this means that the "vertical part" in the generalized derivatives ∂ does not interfere.

3. The min solution operator R_0^t is a finite dimensional "discretization" of the Lax-Oleinik semigroup (groupoid) in weak KAM theory, defined by

$$T_s^t v(x) = \inf_{\gamma(t)=x} \{v(\gamma(s)) + \textstyle\int_s^t L(t, \gamma(t), \dot\gamma(t)) dt\}$$

   where L is the Legendre tranform of H with respect to the p variable, and the inf is taken over all absolutely continuous paths $\gamma : [s, t] \to \mathbb{R}^d$.

Theorem 2.39 (T. Joukovskaia [45])**.** *The min solution $R_0^t v(x)$ is the viscosity solution of the Cauchy problem* (H-J).

Joukovskaia's proof relies on a special characterization of viscosity solutions, valid only for convex Hamiltonians. In the next section, we will give a new proof making sense in more general settings.

In the rest of the section, we will give a slight generalization of Theorem 2.39 from convex Hamiltonians to the convex-concave type Hamiltonians. The minmax in this case has been treated in [12]. Let us consider the Cauchy problem (H-J) with

$$(SP) \qquad H(t, x, p) = H_1(t, x_1, p_1) + H_2(t, x_2, p_2), \quad v(x) = v_1(x_1) + v_2(x_2)$$

where $(x,p) = (x_1, x_2, p_1, p_2) \in T^*\mathbb{R}^d$, and H_1 and H_2 are strictly convex and concave in p respectively. We may assume that each v_i is globally Lipschitz and H_i verifies the condition of finite propagation speed.

Lemma 2.40. *Suppose $S(x,\xi)$ and $S'(y,\eta)$ are two GFQI, then*

$$\inf \max_{(\xi,\eta)} \Big(S(x,\xi) + S'(y,\eta) \Big) = \inf \max_{\xi} S(x,\xi) + \inf \max_{\eta} S'(y,\eta)$$

Proof. Recall the definition of minmax:

$$\inf \max S(x,\xi) = \inf_{[\sigma]\in A} \max_{\xi\in|\sigma|} S(x,\xi)$$

where A is a generator of the homology group $H_{k_\infty}(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2)$ with k_∞ the Morse index of the nondegenerate quadratic form related to S. Similarly we denote A' and k'_∞ the generator and Morse index related to S'. Write

$$\tilde{S}(x,y,\xi,\eta) := S(x,\xi) + S'(y,\eta)$$

Note that we have a homotopy equvalence

$$(\tilde{S}_{x,y}^\infty, \tilde{S}_{x,y}^{-\infty}) = (S_x^\infty, S_x^{-\infty}) \times (S_y^{'\infty}, S_y^{'-\infty})$$

By Künneth formula, since the homology groups for other degrees except k_∞ and k'_∞ are all 0, we have

$$H_{k_\infty + k'_\infty}(\tilde{S}_{x,y}^\infty, \tilde{S}_{x,y}^{-\infty}) \simeq H_{k_\infty}(S_x^\infty, S_x^{-\infty}) \otimes H_{k'_\infty}(S_y^{'\infty}, S_y^{'-\infty})$$

Thus

$$\inf \max \tilde{S} = \inf_{[\tilde{\sigma}]=A\otimes A'} \max_{(\xi,\eta)\in|\tilde{\sigma}|} \tilde{S} \leq \inf_{[\sigma]=A} \max_{\xi\in|\sigma|} S + \inf_{[\sigma']=A'} \max_{\eta\in|\sigma'|} S'$$

since for any descending cycles σ, σ' such that $[\sigma] = A$ and $[\sigma'] = A'$, their product $\sigma \otimes \sigma'$ verifies $[\sigma \otimes \sigma'] = A \otimes A'$.

By the same argument, we have

$$\sup \min \tilde{S} \geq \sup \min S + \sup \min S'$$

using the fact that minmax and maxmin are equal, we get the conclusion

$$\inf \max \tilde{S} = \inf \max S + \inf \max S'$$

□

Lemma 2.41. *Suppose that H and v satisfy (SP), then*

$$R_H^t v(x) = \sum_{i=1}^{2} R_{H_i}^t v_i(x_i).$$

Proof. Let $S_{(H,v)}$ denote the GFQI related to (H,v), i.e. the GFQI of the Lagrangian submanifold $\varphi_H^{0,t}(\partial v)$, then we get a GFQI of $\varphi_H^{0,t}(\partial v)$

$$S_{(H,v)}(t,x,\xi) = \sum_{i=1}^{2} S_{(H_i,v_i)}(t,x_i,\xi_i)$$

where $x = (x_1, x_2)$, and $\xi = (\xi_1, \xi_2)$.

It follows from Lemma 2.40 that the minmax is of splitting form. □

Let J be the viscosity solution operator for the (H-J) equation.

Lemma 2.42. *Suppose that H and v satisfy (SP), then*

$$J_H^t v(x) = \sum_{i=1}^{2} J_{H_i}^t v_i(x_i)$$

Proof. Note that by the stability of viscosity solutions, it is sufficient to prove the statement in the case where each H_i and v_i are smooth. We take the vanishing viscosity argument. Let $u_i^\epsilon(t, x_i)$ be the respective solutions of

$$\begin{cases} \partial_t u_i^\epsilon + H_i(t, x_i, \partial_{x_i} u_i^\epsilon) = \epsilon \Delta u_i^\epsilon \\ u_i^\epsilon(0, x) = v_i(x) \end{cases}, \quad i = 1, 2.$$

It is known that thus regularized solutions converge to the viscosity solution, that is, $u_i^\epsilon(t, x_i) \to J_{H_i}^t v_i(x_i)$ with uniform convergence on compact subsets as $\epsilon \to 0$. On the other hand, $u^\epsilon(t, x) := \sum_{i=1}^2 u_i^\epsilon(t, x_i)$ is the solution of

$$\begin{cases} \partial_t u^\epsilon + H(t, x, \partial_x u^\epsilon) = \epsilon \Delta u^\epsilon, \\ u^\epsilon(0, x) = v(x). \end{cases} \tag{2.3}$$

Since $u^\epsilon(t, x)$ converges to $u(t, x) := \sum_{i=1}^2 J_{H_i}^t v_i(x_i)$ as $\epsilon \to 0$, we get that $u(t, x)$ is the viscosity solution $J_H^t v(x)$. □

Proposition 2.43. *Suppose that H and v satisfy (SP), then the minmax solution $R_H^t v(x)$ is the viscosity solution of the (H-J) equation.*

Proof. Since each H_i is convex or concave in p_i, we get from Theorem 2.39 that the minmax, which, in these cases, reduced to min or max, coincide with the viscosity solution, that is $R_{H_i}^t v_i(x_i) = J_{H_i}^t v_i(x_i)$, hence we can conclude by applying Lemma 2.41 and Lemma 2.42. □

2.4 Iterated minmax and viscosity solution

In contrast with the case of convex Hamiltonians, where the minmax is reduced to a min and provides the viscosity solution, for general non-convex Hamiltonians, the minmax and the viscosity solution may differ: see [63, 65, 12] for counterexamples, and also [24] for a very nice geometric illustration of the fact that the viscosity solution is not necessarily contained in the geometric solution.

Particularly, in [63], the author pointed out without proof that the minmax does not provide a semi-group as a consequence of not being viscosity. We will make this point clear by showing that the semi-group property is a sufficient condition for the minmax to be viscosity.

Proposition 2.44. *Given v, the minmax $R_H^{0,t} v(x)$ is the viscosity solution of the Cauchy problem* (H-J) *if it has the semigroup property with respect to time, that is,*

$$R_H^{0,t} v(x) = R_H^{s,t} \circ R_H^{0,s} v(x), \quad 0 \le s < t \le T.$$

Proof. Suppose $R_0^t v(x) := R_H^{0,t} v(x)$ possesses the semi-group property, we first show that $R_0^t v(x)$ is a viscosity subsolution. For any (t,x), let ψ be a C^2 function such that $\psi(s,y) =: \psi_s(y) \geq R_0^s v(y)$, with equality at (t,x). It is enough to consider ψ in a neighborhood of (t,x), where it has bounded second derivative. Then

$$\psi_t(x) = R_\tau^t \circ R_0^\tau v(x) \leq R_\tau^t \psi_\tau(x). \tag{2.4}$$

By Lemma 2.8, for $t-\tau > 0$ small enough, the characteristics originating from $d\psi_\tau$ do not intersect: let $(x_t, y_t) = \varphi_\tau^t(x_\tau, \partial_x \psi_\tau(x_\tau))$, where φ denotes the Hamiltonian flow of H, then the map $p : (x_\tau, \partial_x\psi_\tau(x_\tau)) \mapsto x_t$ is a diffeomorphism. Therefore $R_\tau^t \psi_\tau(x)$ is a classical C^2 solution of the (H-J) equation. Hence

$$R_\tau^t \psi_\tau(x) = \psi_\tau(x) - \int_\tau^t H(s, x, \partial_x R_\tau^s \psi_\tau(x)) ds \tag{2.5}$$

Moreover, since $(x, \partial_x R_\tau^t \psi_\tau(x)) = \varphi_\tau^t \circ p^{-1}(x)$, we get that $\partial_x R_\tau^t \psi_\tau(x)$ is continuous in τ.

Subtract (2.5) into (2.4), move $\psi_t(x)$ to the RHS, divide both side by $t-\tau$ and let $\tau \to t$, we get

$$0 \leq -\partial_t \psi_t(x) - H(t, x, \partial_x \psi_t(x))$$

from which we get a subsolution by definition. Similarly, we can prove that $R_0^t v(x)$ is a viscosity supersolution.

□

As a direct consequence, we get Theorem 2.39 since the min solutions forms a semi-group (Proposition 2.35).

We remark that Proposition 2.44 does not essentially depend on the variational formulation of the minmax. The following more general statement, containing the minmax as a special case. See for example [9] Prop. 20 and [35] Theorem 5.1.

Proposition 2.45 ([9],[35])**.** *Suppose an operator $T_s^t : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d)$ satisfies the following properties:*

1. *Semi-group: $T_s^t \circ T_\tau^s = T_\tau^t$;*
2. *Monotonicity: $v \geq v' \Rightarrow T_\tau^t v \geq T_\tau^t v'$,*
3. *Compatibility with the (H-J) equation: if v is a C^2 function with bounded derivatives, then there exists an $\epsilon > 0$ such that $T_s^t v$ is a C^2 solution of the (H-J) equation for $|t-s| < \epsilon$.*

then T_0^t is the viscosity solution operator of the (H-J) equation.

Remark 2.46. The presence of rarefaction in the equations of conservation laws in dimension one, where the Hamiltonian H depends only on p, serves as simple examples for the difference between minmax and viscosity solutions, see Example 2.82.

To compensate the defect of the minmax for not being a semi-group, an idea due to M. Chaperon is to replace the "minmax" by some "iterated minmax". Roughly speaking, an iterated minmax is obtained by dividing a given time interval into small pieces and take the minmax step by step. This is a priori a discrete semi-group with respect to the points of the subdivision. We are going to show that, as the steps of the subdivision go to zero, the iterated minmax will converge to a real semi-group, and therefore to the viscosity solution.

In the following, we denote the Lipschitz constant of a global Lipschitz function f by $\|\partial f\|$ and $|\cdot|_K$ denotes the maximum norm on a compact set K.

Lemma 2.47. *Assuming $H \in C_c^2([0,T] \times T^*\mathbb{R}^d)$ and $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$, we have the following estimations:*

1) *$R_H^{s,t}$ defines an operator from $C^{\mathrm{Lip}}(\mathbb{R}^d)$ to $C^{\mathrm{Lip}}(\mathbb{R}^d)$, and*

$$\|\partial(R_H^{s,t}v)\| \le \|\partial v\| + \|\partial_x H\||t-s|$$

2) *For any $0 \le s < t_i \le T$, $i = 1, 2$,*

$$|R_H^{s,t_1}v(x) - R_H^{s,t_2}v(x)| \le |t_1 - t_2| \max_{t \in [t_1,t_2]} |H(t,x,\cdot)|_Y$$

where $Y = \{y : |y| \le \|\partial v\| + \|\partial_x H\| \max_i |t_i - s|\}$.

3) *Let H^0 and H^1 be two Hamiltonians, then*

$$|R_{H^0}^{s,t}v - R_{H^1}^{s,t}v|_{C^0} \le |t-s| \max_{\tau \in [s,t], y \in Y'} |(H^0 - H^1)(\tau, \cdot, y)|_{C^0}$$

where $Y' = \{y : |y| \le \|\partial v\| + \max_i \|\partial_x H^i\||t-s|\}$.

4) *If $v^0, v^1 \in C^{\mathrm{Lip}}(\mathbb{R}^d)$ and K is a compact set in $\mathbb{R}^d$, then there exists a bounded subset $\tilde{K} \subset \mathbb{R}^d$ which depends on $K \times [0,T]$ and the constants $\|\partial v^i\|$, such that*

$$|R_H^{s,t}v^0 - R_H^{s,t}v^1|_K \le |v^0 - v^1|_{\tilde{K}}. \tag{2.6}$$

Proof. The proof is based on Proposition 1.44 with some variation on the original variable x, which can be either $t \in [0,T]$, or $x \in \mathbb{R}^d$ or some parameter λ for the generating family constructed as below.

For simplicity, we may first assume that $|t-s| < \delta_H$ so that

$$S^{s,t}(x, x_0, y_0) = v(x_0) + \phi_H^{s,t}(x, y_0) + xy_0 - x_0 y_0$$

Let $(x(\tau), y(\tau))$ denote the Hamiltonian flow, and $C(x)$ be the critical set defined in Proposition 1.44.

1) For $(x_0, y_0) \in C(x)$, we have

$$\partial_x S^{s,t}(x, x_0, y_0) = \partial_x \phi_H^{s,t}(x, y_0) + y_0 = y(t)$$

where

$$y(t) = y_0 - \int_s^t \partial_x H(\tau, x(\tau), y(\tau)) d\tau, \quad y_0 \in \partial v(x_0)$$

Hence by (1.13),

$$\partial R_H^{s,t} v(x) \subset \mathrm{co}\{y(t),\, y_0 \in \partial v(x_0)\}$$

thus

$$\|\partial(R_H^{s,t}v)\| \le \|\partial v\| + \|\partial_x H\||t-s|.$$

2) For $(x_0, y_0) \in C(x)$, by Lemma 1.16, we have

$$\partial_t S^{s,t}(x, x_0, y_0) = \partial_t \phi_H^{s,t}(x, y_0) = -H(t, x, y(t))$$

Hence

$$\partial_t R_H^{s,t} v(x) \subset \mathrm{co}\{-H(t,x,y(t)),\, y_0 \in \partial v(x_0)\}$$

from which

$$|R_H^{s,t_1}v(x) - R_H^{s,t_2}v(x)| \le |t_1 - t_2| \max_{t \in [t_1,t_2], y \in Y} |H(t,x,y)|$$

where $Y = \{y : |y| \leq \|\partial v\| + \|\partial_x H\| \max_i |t_i - s|\}$.

3) Let $H^\lambda = (1-\lambda)H^0 + \lambda H^1$, $\lambda \in [0,1]$, and let $S_\lambda^{s,t}$ be the corresponding generating families. Fix λ, for (x_0, y_0) in the critical set $C^\lambda(x)$ corresponding to H^λ,

$$\partial_\lambda S_\lambda^{s,t}(x, x_0, y_0) = \partial_\lambda \phi_{H^\lambda}^{s,t}(x, x_0, y_0) = \int_s^t (H^0 - H^1)(\tau, x^\lambda(\tau), y^\lambda(\tau)) d\tau.$$

where the proof of the second equality is similar to that of Lemma 1.16. Hence

$$\partial_\lambda R_{H^\lambda}^{s,t} v(x) \subset \mathrm{co}\{\int_s^t (H^0 - H^1)(\tau, x^\lambda(\tau), y^\lambda(\tau)) d\tau,\ y_0 \in \partial v(x_0)\}$$

from which

$$\begin{aligned} |R_{H^0}^{s,t} v(x) - R_{H^1}^{s,t} v(x)| &\leq \int_0^1 \int_s^t |H^0 - H^1|(\tau, x^\lambda(\tau), y^\lambda(\tau)) d\tau d\lambda \\ &\leq |t-s| \max_{\tau \in [s,t], y \in Y'} |(H^0 - H^1)(\tau, \cdot, y)|_{C^0} \end{aligned}$$

where $Y' = \{|y| : |y| \leq \|\partial v\| + \max_i \|\partial_x H^i\| |t-s|\}$.

4) Let $v^\lambda = (1-\lambda)v^0 + \lambda v^1$, $\lambda \in [0,1]$ and $S_\lambda^{s,t}$ denotes the corresponding generating families, then $\partial_\lambda S_\lambda^{s,t}(x, x_0, y_0) = v^1(x_0) - v^0(x_0)$,

$$\partial_\lambda R_H^{s,t} v^\lambda(x) \subset \mathrm{co}\{v^1(x_0) - v^0(x_0) : (x_0, y_0) \in C^\lambda(x)\}$$

with $C^\lambda(x) \subset \{(x_0, y_0) : |x_0| \leq |x| + T\|\partial_y (H|_{\{y \in Y\}})\|\}$, $Y := \{y : |y| \leq \|\partial v\| + T\|\partial_x H\|\}$. If we take $\tilde{K} = \{x_0 : |x_0| \leq |x|_K + T\|\partial_y (H|_{\{y \in Y\}})\|\}$, we obtain

$$|R_H^{s,t} v^0 - R_H^{s,t} v^1|_K \leq |v^0 - v^1|_{\tilde{K}}$$

In general, the above results follow from the fact that the critical set $C(x)$ defines the Hamiltonian flow $(x(\tau), y(\tau))_{s \leq \tau \leq t}$ for any $0 \leq s < t \leq T$. □

Remark 2.48. The estimates in the proposition, more subtle than needed, precisely reveal that being with finite propagation speed is enough to define the minmax function.

Now given any compact subset $K \subset \mathbb{R}^d$, we consider $(t, x) \in [0, T] \times K$.

Given a subdivision $\zeta_n = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0, T]$, for each $s \in [0, T]$, we associate to it a number $m(\zeta_n, s)$, depending on ζ_n :

$$m(\zeta_n, s) := i, \quad \text{if} \quad t_i \leq s < t_{i+1}.$$

For simplicity, fixing a subdivision, we may abbreviate $m(\zeta_n, s)$ as $m(n, s)$.

Definition 2.49. The *iterated minmax solution operator* for the (H-J) equation with respect to a subdivision ζ_n is defined as follows: for $0 \leq s' < s \leq T$,

$$R_{H,\zeta_n}^{s',s} := R_H^{t_{m(n,s)},s} \circ \cdots \circ R_H^{s',t_{m(n,s')+1}}.$$

When the Hamiltonian H is fixed , we may abbreviate our notation $R_H^{s,t}$ as R_s^t, and the iterated minmax as

$$R_{s',n}^s := R_{t_{m(n,s)}}^s \circ \ldots R_{s'}^{t_{m(n,s')+1}}. \tag{2.7}$$

for which we call it a n-step minmax, with a subdivision indicated.

Define the length of ζ_n by $|\zeta_n| := \max_i |t_i - t_{i+1}|$. Suppose that$\{\zeta_n\}_n$ is a sequence of subdivisions of $[0,T]$ such that $|\zeta_n|$ tends to zero as n goes to infinity, and $\{R^s_{0,n}v(x)\}_n$ be the corresponding sequence of iterated minmax solutions for an initial function $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$.

Lemma 2.50. *The sequence of functions $u_n(s,x) := R^s_{0,n}v(x)$ is equi-Lipschitz and uniformly bounded for $(s,x) \in [0,T] \times K$.*

Proof. By Lemma 2.47, one can verify that

$$\begin{aligned}&\|\partial(R^s_{0,n}v)\| \le \|\partial v\| + T\|\partial_x H\|,\\ &|R^s_{0,n}v - R^t_{0,n}v|_K \le |H|_{\mathcal{K}}|s-t|, \qquad s,t \in [0,T].\end{aligned}$$

where $\mathcal{K} := \{(t,x,y) : t \in [0,T], x \in K, |y| \le \|\partial v\| + T\|\partial_x H\|\}$.

In particular, taking $t = 0$, we get

$$|R^s_{0,n}v|_K \le |v|_K + T|H|_{\mathcal{K}}, \qquad s \in [0,T]$$

□

It follows immediately from the Arzela-Ascoli Theorem that $\{R^s_{0,n}v(x)\}_n$ has uniformly convergent subsequences on $[0,T] \times K$. Fixing a convergent subsequence $\{R^s_{0,n_k}v(x)\}_k$, we write its limit as

$$\bar{R}^s_0 v(x) := \lim_{k\to\infty} R^s_{0,n_k}v(x).$$

Remember that $\bar{R}^s_0 v(x)$ depends a priori on the specified subsequence of subdivisions $\{\zeta_{n_k}\}_k$, which itself depends on the given initial function v and the given subdivisions $\{\zeta_n\}_n$. We define the related limit operator for R^s_{s',n_k} with respect to the fixed subsequence of subdivisions $\{\zeta_{n_k}\}_k$: for any time $0 \le s' < s \le T$,

$$\bar{R}^s_{s'} := \lim_{k\to\infty} R^s_{s',n_k}, \qquad \text{if the limit exists}$$

Lemma 2.51. *We have*

$$\bar{R}^s_0 v(x) = \lim_{k\to\infty} R^s_{s',n_k} \circ \bar{R}^{s'}_0 v(x) = \bar{R}^s_{s'} \circ \bar{R}^{s'}_0 v(x), \quad \forall\, 0 \le s' < s \le T. \tag{2.8}$$

Proof. For brevity, we omit the subindex k of n_k.

We first claim that

$$\bar{R}^s_0 v(x) = \lim_{n\to\infty} R^{t_{m(n,s)}}_{0,n} v(x) \tag{2.9}$$

Indeed,

$$\begin{aligned}|R^s_{0,n}v(x) - R^{t_{m(n,s)}}_{0,n}v(x)| &= |R^s_{t_{m(n,s)}} \circ R^{t_{m(n,s)}}_{0,n}v(x) - R^{t_{m(n,s)}}_{0,n}v(x)|\\ &\le |H|_{\mathcal{K}}(s - t_{m(n,s)}) \le |H|_{\mathcal{K}}|\zeta_n| \to 0, \quad n \to \infty.\end{aligned}$$

Assume that the uniform convergence of $\{R^{t_{m(n,s)}}_0 v(x)\}_n$ on a bit larger set $\tilde{K} \times [0,T]$ where $\tilde{K} \supset K$ as defined in 4) of Lemma 2.47. Then for any $\epsilon > 0$, there exists N large enough such that for any $i,j > N$,

$$|R^{t_{m(i,s)}}_{0,i} v - R^{t_{m(j,s)}}_{0,j} v|_{\tilde{K}} < \epsilon, \quad \forall s \in [0,T]$$

Hence

$$\begin{aligned}|R^{t_{m(i,s)}}_{t_{m(i,s')},i} \circ R^{t_{m(j,s')}}_{0,j} v - R^{t_{m(i,s)}}_{0,i} v|_K &= |R^{t_{m(i,s)}}_{t_{m(i,s')},i} \circ R^{t_{m(j,s')}}_{0,j} v - R^{t_{m(i,s)}}_{t_{m(i,s')},i} \circ R^{t_{m(i,s')}}_{0,i} v|_K \\ &\leq |R^{t_{m(j,s')}}_{0,j} v - R^{t_{m(i,s')}}_{0,i} v|_{\tilde{K}} < \epsilon.\end{aligned}$$

Let j go to infinity, we get

$$|R^{t_{m(i,s)}}_{t_{m(i,s')},i} \circ \bar{R}^{s'}_0 v - R^{t_{m(i,s)}}_{0,i} v|_K < \epsilon, \quad i > N$$

Thus

$$\bar{R}^s_0 v(x) = \lim_{i\to\infty} R^{t_{m(i,s)}}_{t_{m(i,s')},i} \circ \bar{R}^{s'}_0 v(x).$$

We conclude by verifying the following, which is similar to (2.9),

$$\lim_{i\to\infty} R^s_{s',i} \circ \bar{R}^{s'}_0 v(x) = \lim_{i\to\infty} R^{t_{m(i,s)}}_{t_{m(i,s')},i} \circ \bar{R}^{s'}_0 v(x).$$

□

Proposition 2.52. *$\bar{R}^s_0 v(x)$ is the viscosity solution of the (H-J) problem.*

Proof. We first show that it is a viscosity subsolution. For any (t, x), suppose ψ a C^2 function defined in a neighborhood of (t, x), having bounded second derivative and such that $\psi(s, y) =: \psi_s(y) \geq \bar{R}^s_0 v(y)$, with equality at (t, x),

$$\psi_t(x) = \bar{R}^t_0 v(x) = \lim_{k\to\infty} R^t_{\tau,n_k} \circ \bar{R}^\tau_0 v(x) \leq \lim_{k\to\infty} R^t_{\tau,n_k} \psi_\tau(x) = R^t_\tau \psi_\tau(x) \tag{2.10}$$

the last equality holds for $t - \tau$ small enough, where the characteristics originating from $d\psi_\tau$ do not intersect, hence the iterated minmax is nothing but the 1-step minmax which is the classical C^2 solution. We conclude by applying the same argument in Proposition 2.44: we have

$$R^t_\tau \psi_\tau(x) = \psi_\tau(x) - \int_\tau^t H(s, x, \partial_x R^s_\tau \psi_\tau(x))ds \tag{2.11}$$

Subtract (2.11) into (2.10), move $\psi_t(x)$ to the RHS, divide both side by $t - \tau$ and let $\tau \to t$, we get

$$0 \leq -\partial_t \psi_t(x) - H(t, x, \partial_x \psi_t(x))$$

from which we get a subsolution by definition. The proof that $\bar{R}^s_0 v(x)$ is a supersolution is similar. □

For given H and v, we say that *the limit of iterated minmax solutions exists* in $[s, t]$, if for any sequence of subdivision $\{\zeta_n\}_{n\in\mathbb{N}}$ of $[s, t]$ such that $|\zeta_n| \to 0$ as $n \to \infty$, the related sequence of iterated minmax solutions $\{R^{s,\tau}_{H,\zeta_n} v(x)\}_{n\in\mathbb{N}}$, $(\tau, x) \in [s, t] \times \mathbb{R}^d$ converges uniformly on compact subsets to a limit which is independent of the choice of subdivisions, then, without ambiguity, we denote this limit also by $\bar{R}^{s,\tau}_H v(x)$. As before, in the case where H is specified once and for all, we may write the iterated minmax solution and its limit by $R^\tau_{s,n} v(x)$ and $\bar{R}^\tau_s v(x)$ respectively.

We can now prove our main Theorem

Theorem 2.53. *Suppose $H \in C^2_c([0, T] \times T^*\mathbb{R}^d)$ and $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$, then for the Cauchy problem of the Hamilton-Jacobi equation*

$$\begin{cases} \partial_t u + H(t, x, \partial_x u) = 0, & t \in (0, T] \\ u(x, 0) = v(x), & x \in \mathbb{R}^d. \end{cases}$$

the limit of iterated minmax solutions exists and coincides with the viscosity solution.

Proof. For any compact subset K and $(t,x) \in [0,T] \times K$, and given any sequence of subdivisions $\{\zeta_n\}_n$, setting $u_n(t,x) = R^t_{0,\zeta_n} v(x)$, we have proved in Proposition 2.52 that any convergent subsequence of $\{u_n\}_n$ converges uniformly on $[0,T] \times K$ to the viscosity solution. Now, by Lemma 2.50 and the Arzela-Ascoli theorem, the sequence of functions u_n takes its values in a compact subset of $C^0([0,T] \times K)$, hence it converges to the viscosity solution. □

It turns out that the minmax selector behaves like a "generator" defined in by P.E. Souganidis in [58], and our limiting iterated minmax procedure fits into his general approximation schemes. The virtue of the iterated minmax approximation, due to its geometric property, is that it may provide us a geometric description of the viscosity solution.

As an application of the limiting iterated minmax method, let us consider the nonhomogeneous (H-J) equation

$$\begin{cases} \partial_t u + H(\partial_x u) = G(t,x), & t \in (0,T] \\ u(0,x) = v(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.12}$$

We may assume that $H : \mathbb{R}^d \to \mathbb{R}$ and $G : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are C^2 with compact supports. Our aim is to construct solutions of (2.12) from solutions of the homogeneous (H-J) equation:

$$\partial_t u + H(\partial_x u) = 0, \quad u(0,x) = v(x). \tag{2.13}$$

Let $E(t,s) : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d)$ denote the operator defined by

$$E(t,s)v(x) = v(x) + (t-s)G(s,x),$$

and $J(t,s) : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d)$ denote the viscosity solution operator for

$$\partial_t u + H(\partial_x u) = 0, \quad u(s,x) = v(x)$$

Theorem 2.54 ([44])**.** *Suppose that $G \in C^1([0,T] \times \mathbb{R}^d)$. Given a subdivision $\zeta_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0,T]$, let*

$$u^N(t,x) = J(t,t_{i-1})E(t_i,t_{i-1})u(t_{i-1},\cdot)(x) + (t-t_i)G(t_{i-1},x), \quad \textit{for } t \in [t_{i-1},t_i]$$

and $u(0,x) = v(x)$. Then

$$|\bar{u}(t,\cdot) - u^N(t,\cdot)| \le K|\zeta_N|, \quad t \in [0,T]$$

where $\bar{u}(t,x)$ is the viscosity solution of (2.12)*.*

The Theorem was given in [44] for more general H. In the following, we will try to prove it using Theorem 2.53 and more properties of the minmax solutions. We denote by $R^G(t,s), R(t,s) : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d)$ the minmax solution operators for

$$\partial_t u + H(\partial_x u) = G(t,x), \quad u(s,x) = v(x) \tag{2.14}$$
$$\partial_t u + H(\partial_x u) = 0, \quad u(s,x) = v(x) \tag{2.15}$$

respectively.

Lemma 2.55. *Suppose that $w \in C^1([0,T]\times\mathbb{R}^d)$, and $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$, then for any $0 \le s < t \le T$,*

$$u(\tau,x) := R(\tau,s)v(x) - w(\tau,x), \quad \tau \in [s,t]$$

is the minmax solution for the (H-J) equation

$$\begin{cases} \partial_\tau u + H(\partial_x u + \partial_x w) = -\partial_\tau w, & \tau \in (s,t] \\ u(s,x) = v(x) - w(s,x). \end{cases} \tag{2.16}$$

Proof. First suppose v and w are C^2, and denote $w_s := w(s,\cdot)$. Let L_u be the geometric solution of (2.16), by definition, it is the Lagrangian submanifold in $T^*(\mathbb{R}\times\mathbb{R}^d)$ which contains the initial submanifold

$$\Gamma(u_s) = \{(s,x,-H(\partial_x v) - \partial_\tau w_s, \partial_x v - \partial_x w_s)\}$$

and is contained in the hypersurface

$$\{(\tau,x,e,p)|e + \partial_\tau w + H(p + \partial_x w) = 0\}$$

Denote $L_w = \{(\tau,x,\partial_\tau w,\partial_x w)\}$ the 1-graph corresponding to w. Let

$$L_u \sharp L_w := \{(\tau,x,e+\partial_\tau w, p+\partial_x w), \, (\tau,x,e,p)\in L_u\}$$

Then $L_u\sharp L_w$ is also a Lagrangian submanifold in $T^*(\mathbb{R}\times\mathbb{R}^d)$, and

$$\Gamma(v) = \{(s,x,-H(\partial_x v),\partial_x v)\} \subset L_u\sharp L_w \subset \{(\tau,x,e',p')|e' + H(p') = 0\}$$

Hence $L_u\sharp L_w$ is the geometric solution of

$$\begin{cases} \partial_\tau u + H(\partial_x u) = 0, \\ u(s,x) = v(x). \end{cases} \tag{2.17}$$

Note that $w(t,x)$ is a generating family of L_w, if $S(\tau,x,\eta)$ is a GFQI of L_u, then

$$(S\sharp w)(\tau,x,\eta) := S(\tau,x,\eta) + w(\tau,x)$$

is a GFQI for $L_u\sharp L_w$. Since the minmax does not depend on the choice GFQI, we get

$$R(\tau,s)v(x) := \inf\max(S\sharp w)(\tau,x,\eta) = \inf\max S(\tau,x,\eta) + w(\tau,x)$$

Hence,

$$u(\tau,x) := \inf\max S(\tau,x,\eta) = R(s,\tau)v(x) - w(\tau,x)$$

is the minmax solution of (2.16).

The result for $v \in C^{\mathrm{Lip}}$ and $w \in C^1$ follows directly from the stability of minmax.

□

Lemma 2.56. *Suppose that $G \in C^2_c([0,T]\times\mathbb{R}^d)$, and $v, v' \in C^{\mathrm{Lip}}(\mathbb{R}^d)$. For any $0 \le s < t \le T$, let*

$$u(\tau,x) = R(\tau,s)E(t,s)v(x) + (\tau - t)G(s,x), \quad \tau \in [s,t]$$

then

$$|u(\tau,\cdot) - R^G(\tau,s)v'|_{C^0} \le |v - v'|_{C^0} + C|t-s||\tau - s|, \quad \tau \in [s,t]$$

where C is a constant depending on H and G.

Proof. By Lemma 2.55, $u(\tau, x)$ is the minmax solution of

$$\begin{cases} \partial_\tau u + H(\partial_x u + (t-\tau)\partial_x G(s,x)) = G(s,x), & \tau \in (s,t] \\ u(s,x) = v(x). \end{cases} \tag{2.18}$$

Then, using the estimates 4) and 3) in Lemma 2.47 for v, v' and $H^0(\tau, x, y) := H(y) - G(\tau, x)$, $H^1(\tau, x, y) := H(y + (t-\tau)\partial_x G(s,x)) - G(s,x)$, we get

$$\begin{aligned} &|u(\tau,x) - R^G(\tau,s)v'(x)| \\ &\le |v - v'|_{C^0} + |\tau - s| \max_{\tau\in[s,t]} |H(y) - G(\tau,x) - (H(y+(t-\tau)\partial_x G(s,x)) - G(s,x))|_{C^0} \\ &\le |v - v'|_{C^0} + |\tau - s| \max_{\tau\in[s,t]} (\|\partial_x H\| \|\partial_x G\| |\tau - t| + \|\partial_t G\| |\tau - s|) \\ &\le |v - v'|_{C^0} + C|t-s||\tau - s|. \end{aligned}$$

where $C = \max\{\|\partial_x H\|\|\partial_x G\|, \|\partial_t G\|\}$. □

Now given a sequence of subdivisions of $[0,T]$, let $R_k^G(t,0)v$ denote the related k-step iterated minmax solutions for the (H-J) equation (2.12), and $\bar{R}^G(t,0)v$ be the limit of some convergence subsequence of $\{R_k^G(t,0)v(x)\}_k$. Let $R_k(t,s)$ be the related k-step iterated minmax solution operator for the equation (2.15), and

$$\bar{R}(t,s) : C^{\mathrm{Lip}}(\mathbb{R}^d) \to C^{\mathrm{Lip}}(\mathbb{R}^d)$$

denote the limit operator of $\{R_k(t,s)\}_k$ which, by Theorem 2.53, is the viscosity solution operator $J(s,t)$ of the homogeneous equation.

For simplicity, given any $N \in \mathbb{N}$, we take the subdivision of $[0,T]$ which divides it into N equal pieces, denoting $t_i = (i/N)T$, $0 \le i \le N$. Then

$$u^N(t,x) = \bar{R}(t,t_{i-1})E(t_i,t_{i-1})u^N(t_{i-1},\cdot)(x) + (t - t_i)G(t_{i-1},x), \quad \text{for } t \in [t_{i-1}, t_i].$$

with $u^N(0,x) = v(x)$.

Proposition 2.57. *Suppose that $G \in C_c^2([0,T]\times\mathbb{R}^d)$ and $v \in C^{\mathrm{Lip}}(\mathbb{R}^d)$, then*

$$|u^N(t,\cdot) - \bar{R}^G(t,0)v|_{C^0} \le \frac{C}{N}.$$

Proof. Let $\{t_0 < \bar{t} < t_1\}$ be any 2-step subdivision of $[t_0, t_1]$, and $R_2(t,t_0)$, $R_2^G(t,t_0)$ the corresponding 2-step minmax operator. Define

$$u_2(t,x) = R_2(t,t_0)E(t_1,t_0)v(x) + (t-t_1)G(t_0,x)$$

By Lemma 2.56, we have, for $t \in [0, \bar{t}]$,

$$|u_2(t,\cdot) - R^G(t,0)v|_{C^0} \le C|t - t_0||t_0 - t_1|$$

In particular,

$$|u_2(\bar{t},\cdot) - R^G(\bar{t},0)v|_{C^0} \le C|\bar{t} - t_0||t_0 - t_1|.$$

For $t \in [\bar{t}, t_1]$,

$$u_2(t,x) = R(t,\bar{t})R(\bar{t},t_0)E(t_1,t_0)v(x) + (t-t_1)G(t_0,x)$$

by Lemma 2.55, it is the minmax solution of

$$\begin{cases} \partial_t u + H(\partial_x u + (t_1 - t)\partial_x G(t_0, x)) = G(t_0, x), & t \in (\bar{t}, t_1] \\ u(\bar{t}, x) = R(\bar{t}, t_0)E(t_1, t_0)v(x) + (\bar{t} - t_1)G(t_0, x) = u_2(\bar{t}, x) \end{cases}$$

Similar to Lemma 2.56, we can get

$$\begin{aligned} |u_2(t, \cdot) - R^G(t, \bar{t})R^G(\bar{t}, t_0)v|_{C^0} &\leq |u_2(\bar{t}, \cdot) - R^G(\bar{t}, t_0)v|_{C^0} + C|t - \bar{t}||t_0 - t_1| \\ &\leq C(|\bar{t} - t_0| + |\bar{t} - t|)|t_0 - t_1| \\ &\leq C|t - t_0||t_0 - t_1|, \qquad t \in [\bar{t}, t_1] \end{aligned}$$

Thus we have proved,

$$|u_2(t, \cdot) - R_2^G(t, t_0)v|_{C^0} \leq C|t - t_0||t_0 - t_1|, \quad t \in [t_0, t_1]$$

By the same arguments, we can show that, for any k-step subdivision of $[t_0, t_1]$,

$$|u_k(t, \cdot) - R_k^G(t, t_0)v|_{C^0} \leq C|t - t_0||t_0 - t_1|, \quad t \in [t_0, t_1]$$

where $u_k(t, x) := R_k(t, t_0)E(t_1, t_0)v(x) + (t - t_1)G(t_0, x)$. Hence taking the limit with respect to the convergent subsequence of $\{R_k^G(t, 0)v\}_k$, we get

$$|u^N(t, \cdot) - \bar{R}^G(t, t_0)v|_{C^0} \leq C|t - t_0||t_0 - t_1|, \quad t \in [t_0, t_1]$$

In particular, we have

$$|u^N(t_1, \cdot) - \bar{R}^G(t_1, t_0)v|_{C^0} \leq C|t_1 - t_0|^2 = \frac{C}{N^2}$$

For $t \in [t_1, t_2]$,

$$\begin{aligned} |u^N(t, \cdot) - \bar{R}^G(t, t_0)v|_{C^0} &= |\bar{R}(t, t_1)E(t_2, t_1)u^N(t_1, \cdot)(x) + (t - t_2)G(t_1, x) - \bar{R}^G(t, t_1)\bar{R}^G(t_1, t_0)v|_{C^0} \\ &\leq |u^N(t_1, \cdot) - \bar{R}^G(t_1, t_0)v|_{C^0} + C|t - t_1||t_1 - t_2| \\ &\leq \frac{C}{N^2} + \frac{C}{N^2} = 2\frac{C}{N^2}, \end{aligned}$$

By induction, we conclude

$$|u^N(t, \cdot) - \bar{R}^G(t, t_0)v|_{C^0} \leq \frac{C}{N}, \quad t \in [0, T]$$

□

Thus Theorem 2.54 is proved using again Theorem 2.53 which ensures that the limit of iterated minmax $\bar{R}^G(t, 0)v(x)$ is viscosity.

2.5 Equations of conservation law in dimension one

Consider the Cauchy problem of Hamiltonian-Jacobi equations, which are related to equations of conservation law, in dimension one:

$$(CL) \begin{cases} \partial_t u + H(\partial_x u) = 0, & t \in (0, T] \\ u(0, x) = v(x), & x \in \mathbb{R} \end{cases}$$

In the following, we assume that $v : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, and $H : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz. This allows defining the minmax solution operator by first truncating H outside a neighborhood of $\mathcal{K} := \{y : |y| \leq \operatorname{Lip} v\}$ so that H has compact support (recall that the minmax so obtained is independant of the truncation). In the sequel, H denotes the truncated Hamiltonian.

Lemma 2.58. *The minmax solution operator $R_H^{s,t} : C^{\mathrm{Lip}}(\mathbb{R}) \to C^{\mathrm{Lip}}(\mathbb{R})$ for the (CL) equation is defined by*

$$R_H^{s,t} v(x) := \inf_{\sigma} \max_{(x_0,y_0)\in\sigma} \Big(v(x_0) - (t-s)H(y_0) + (x-x_0)y_0 \Big),$$

Proof. First suppose that $H \in C^2$, then the Hamiltonian flow of H is:

$$\varphi^{s,t}(x_0, y_0) = (x_0 + (t-s)H'(y_0), y_0)$$

It admits a generating function $\phi^{s,t}(x, y_0) = -(t-s)H(y_0)$, thus a generating family $\varphi^{s,t}(\partial v)$ is given by

$$S(x, x_0, y_0) = v(x_0) - (t-s)H(y_0) + (x-x_0)y_0$$

In general when H is Lipschitz, S is also well-defined, and so is the minmax. □

It is worth noticing that since H does not depend on t, the minmax operator $R_H^{s,t}$ could be written as R_H^{t-s}, which depends only on the length of passing time $t-s$.

For H depending only on p, the estimates for the minmax operator in Lemma 2.47 can be simplified. In particular, we remark that the Lipschitz constant of the minmax solution is always bounded by that of the initial function. We summarize them in the following.

Proposition 2.59. *Assume that $H : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and $v \in C^{\mathrm{Lip}}(\mathbb{R})$,*

1) *The Lipschitz constant satisfies $\|\partial(R_H^{s,t} v)\| \le \|\partial v\|$;*

2) *For $t_1, t_2 \ge 0$,*

$$|R_H^{s,t_1} v(x) - R_H^{s,t_2} v(x)| \le |t_1 - t_2| |H|_{\{|p| \le \|\partial v\|\}}.$$

3) *Let H^0 and H^1 be two Hamiltonians, then*

$$|R_{H^0}^{s,t} v - R_{H^1}^{s,t} v|_{C^0} \le |t-s| |H^0 - H^1|_{\{|p| \le \|\partial v\|\}}.$$

4) *If $v^0, v^1 \in C^{\mathrm{Lip}}(\mathbb{R})$ and K is a compact set in $\mathbb{R}$, then*

$$|R_H^{s,t} v^0 - R_H^{s,t} v^1|_K \le |v^0 - v^1|_{\tilde{K}}.$$

Lemma 2.60. *Suppose that $t > 0$, and $\mathcal{O} \subset \mathbb{R}$ is an open subset. If for every $\epsilon > 0$, there exists $s \in (0, \epsilon)$ such that $R_s^\tau \circ R_0^s v(x)$ is a viscosity solution of the (CL) equation for $(\tau, x) \in (s, t) \times \mathcal{O}$, then $R_0^\tau v(x)$ is also a viscosity solution for $(\tau, x) \in (0, t) \times \mathcal{O}$.*

Proof. We will apply the standard argument for the stability of viscosity solutions. Let $s_n > 0$ be a sequence of decreasing numbers such that $s_n \to 0$, denote

$$u_n(\tau, x) := R_{s_n}^\tau \circ R_0^{s_n} v(x), \quad (\tau, x) \in \mathcal{O}_{s_n} := (s_n, t) \times \mathcal{O}$$

and $u(\tau, x) := R_0^\tau v(x)$.

Given any $\bar{y} := (\bar{\tau}, \bar{x}) \in \mathcal{O}_0$, let $\psi \in C^1(\mathcal{O}_{\bar{\tau}/2})$ and $u - \psi$ attain a local maximum at $(\bar{\tau}, \bar{x})$. Take $\tilde{\psi} \in C^1(\mathcal{O}_{\bar{\tau}/2})$ such that $0 \le \tilde{\psi} < 1$ if $(\tau, x) =: y \ne \bar{y}$ and $\tilde{\psi}(\bar{y}) = 1$. Then $u - (\psi - \tilde{\psi})$ attains a strict local maximum at $\bar{y}$. For n large enough, u_n is well-defined in $\mathcal{O}_{\bar{\tau}/2}$, and by 4) in Proposition 2.59, $u_n \to u$ on $\overline{\mathcal{O}_{\bar{\tau}/2}}$, thus there exists $y_n \in \mathcal{O}_{\bar{\tau}/2}$ such that $u_n - (\psi - \tilde{\psi})$ attains a local maximum at y_n and $y_n \to \bar{y}$. By assumption, u_n are viscosity solutions, hence

$$\partial_t(\psi - \tilde{\psi})(y_n) + H(\partial_x(\psi - \tilde{\psi})(y_n)) \leq 0$$

and we conclude that

$$\partial_t\psi(\bar{y}) + H(\partial_x\psi(\bar{y})) \leq 0$$

since $d\tilde{\psi}(\bar{y}) = (\partial_t\tilde{\psi}, \partial_x\tilde{\psi})(\bar{y}) = 0$. Thus we have proved that u is a viscosity subsolution. Similarly, we can prove that it is a viscosity supersolution, hence a viscosity solution. □

2.5.1 Hopf formula for convex initial functions

We recall that, if the Hamiltonian $H(p)$ is convex or concave, the viscosity solution is given by the so-called Hopf formula (ref. Example 2.34). In the literature, there is also another version of Hopf formula related to convex or concave initial functions.

In the following, we say $f : \mathbb{R} \to \mathbb{R}$ is a convex function if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \quad \lambda \in [0,1].$$

We dot not require f to be C^2 with positively definite Hessian (i.e. strictly convex). Denote

$$f^* : \mathbb{R} \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}, \quad f^*(x) := \sup_y (xy - f(y))$$

the convex conjugate of any extended function $f : \mathbb{R} \to \overline{\mathbb{R}}$ not necessarily convex. This is a generalization of the Legendre transform which is usually defined for strictly convex functions.

Remark 2.61. The convention of adding $+\infty$ to the image domain of f^* reduces the task of discussing $\mathrm{dom}(f^*) = \{x : f^*(x) < \infty\}$. Indeed, if f is a Lipschitz function, one can show that

$$\mathrm{dom}(f^*) = \overline{\{y : \exists x,\, y \in \partial f(x)\}}$$

See for example [61]. In particular, if f is globally Lipschitz, then $\mathrm{dom}(f^*) \subset \{y : |y| \leq \mathrm{Lip}(f)\}$. On the contrary, in the classical Legendre transform, $\mathrm{dom}(f^*) = \mathbb{R}^d$.

Given $H \in C(\mathbb{R})$, for any convex initial function $v \in C^{\mathrm{Lip}}(\mathbb{R})$, the Hopf formula for the (CL) equation is

$$u(t,x) = (v^* + tH)^*(x) = \max_{y_0}(xy_0 - (v^*(y_0) + tH(y_0)))$$

Proposition 2.62 (Proposition 1,[49]). *If v is convex, the Hopf formula defines a function which coincides with the viscosity solution of the (CL) equation .*

As before, we denote the viscosity solution by $J_0^t v(x)$, hence

$$J_0^t v(x) = \max_{y_0}(xy_0 - (v^*(y_0) + tH(y_0)))$$

Remark 2.63. One can also define the concave conjugate for extended functions $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ by

$$f_*(x) = \inf_y (xy - f(y)).$$

It possesses properties similar to those of the convex conjugate. Especially, $f_{**} = f^{\frown}$ is the upper concave envelop provided f is upper semi-continuous.

Lemma 2.64 ([61])**.** *Let $v : \mathbb{R} \to \mathbb{R}$ be a convex Lipschitz function, and $v^* : \mathbb{R} \to \overline{\mathbb{R}}$ be its convex conjugate, then*

$$y \in \partial v(x) \Longleftrightarrow v^*(y) = xy - v(x) \Longleftrightarrow x \in \partial v^*(y).$$

Proof. Since v is convex, its generalized derivative $\partial v(x)$ is the usual sub derivative,

$$\begin{aligned} \partial v(x) &= \{y : \forall x', v(x') \geq v(x) + y(x' - x)\} \\ &= \{y : \forall x', xy - v(x) \geq x'y - v(x')\}. \end{aligned}$$

We conclude by the definition of the convex conjugate $v^*(y) = \max_{x'} x'y - v(x')$. Replace v by v^* in the above argument, we get that $x \in \partial v^*(y)$ if and only if $v(x) = xy - v^*(y)$. □

Recall that the wavefront of v is defined by

$$\mathcal{F}^t = \mathcal{F}^t(v) := \{(x, S_t(x, x_0, y_0)) | y_0 \in \partial v(x_0), x \in x_0 + t\partial H(y_0)\}.$$

Lemma 2.65. *If v is a convex Lipschitz function, we have the relation*

$$R_0^t v(x) \leq \max\{u : (x, u) \in \mathcal{F}^t(v)\} \leq J_0^t v(x).$$

Proof. Recall that a generating family for the minmax is given by

$$\begin{aligned} S_t(x, x_0, y_0) &= v(x_0) + xy_0 - tH(y_0) - x_0 y_0 \\ &= (v(x_0) - x_0 y_0) + xy_0 - tH(y_0). \end{aligned}$$

By lemma 2.64

$$y_0 \in \partial v(x_0) \Leftrightarrow v^*(y_0) = x_0 y_0 - v(x_0),$$

hence

$$\mathcal{F}^t(v) \subset \{(x, u) : u = xy_0 - v^*(y_0) - x_0 y_0\}.$$

The second inequatlity then follows from the Hopf formula.

Since the minmax is a critical value for the map $(x_0, y_0) \mapsto S_t(x, x_0, y_0)$, the graph of $R_0^t(v)$ is contained in $\mathcal{F}^t(v)$, hence the first inequality.

□

In the following, we will show in a concrete way that the minmax and the viscosity solution for a convex initial function are indeed the same.

Let us consider the (linear) Riemann problem, which has initial functions of the form

$$v(x) = \begin{cases} p_- x, & x \leq 0, \\ p_+ x, & x \geq 0. \end{cases}$$

There are two possibilities: if $p_- < p_+$, then v is convex; if $p_- > p_+$, v is concave. We take for example the convex case. Note that $v^* = 0$ in $\mathrm{dom}(v^*) = [p_-, p_+]$. Let

$$H_v(p) = \begin{cases} H(p), & p \in \mathrm{dom}(v^*), \\ \infty, & \text{otherwise}. \end{cases}$$

Denote the lower convex envelope of H_v by $H_v^{\smile}$,

$$H_v^{\smile}(p) = H_v^{**}(p) = \begin{cases} (H|_{\mathrm{dom}(v^*)})^{**}(p), & p \in \mathrm{dom}(v^*), \\ \infty, & \text{otherwise}, \end{cases}$$

where

$$(H|_{\mathrm{dom}(v^*)})^{**}(p) = \sup\{h(p) | h \leq H, \, h \,\mathrm{convex\,on\,dom}(v^*)\}$$

is the lower convex envelop of H on $\mathrm{dom}(v^*)$.

Lemma 2.66. *For the Riemann problem with convex initial value v, we have*

$$J_0^t v(x) = J_{H_v^{\smile}}^{0,t} v(x) = (tH_v^{\smile})^*(x).$$

Proof.

$$\begin{aligned}
J_0^t v(x) &= (v^* + tH)^*(x) = \max_{y_0 \in \mathrm{dom}(v^*)} (xy_0 - v^*(y_0) - tH(y_0)) \\
&= \max_{y_0 \in \mathrm{dom}(v^*)} (xy_0 - tH(y_0)) = \max_{y_0} (xy_0 - tH_v(y_0)) \\
&= (tH_v)^*(x) = tH_v^*(\frac{x}{t}) = tH_v^{***}(\frac{x}{t}) \\
&= (tH_v^{**})^*(x) = (tH_v^{\smile})^*(x)
\end{aligned}$$

□

Now we can give an explicit description of the viscosity solution of the Riemann problem, with piecewise linear Hamiltonian H.

By breakpoints of a piecewise linear function, we refer to the points where the function is not C^1.

Lemma 2.67 ([30])**.** *For the Riemann problem with H piecewise linear such that $(H|_{[p_-,p_+]})^{\smile}$ (resp. the concave envelop $(H|_{[p_+,p_-]})^{\frown}$) in has m breakpoints in (p_-, p_+) (resp. (p_+, p_-)), then the viscosity solution $J_0^t v(x)$ has $m+1$ shocks*[4] *with constant speed originated from the origin.*

Proof. We take the case where $p_- < p_+$ for example. Let $p_1 < p_2 < \cdots < p_m$ denote the break points of $(H|_{[p_-,p_+]})^{\smile}$ in (p_-, p_+), and $p_0 = p_-$, $p_{m+1} = p_+$. Set

$$s_i = \frac{H(p_{i+1}) - H(p_i)}{p_{i+1} - p_i}, \quad 0 \le i \le m$$

Since $H^{\smile}$ is convex, we have $s_i < s_{i+1}$.

$$J_0^t v(x) = \max_{y_0 \in [p_-,p_+]} (xy_0 - tH^{\smile}(y_0)) = \max_{0 \le i \le m} \max_{y_0 \in [p_i,p_{i+1}]} (x - ts_i)y_0 + t(s_i p_i - H(p_i))$$

An easy calculation gives us that

$$J_0^t v(x) = \begin{cases} xp_0 - tH(p_0), & x \le ts_0 \\ xp_{i+1} - tH(p_{i+1}), & x \in [ts_i, ts_{i+1}], 0 \le i \le m-1 \\ xp_{m+1} - tH(p_{m+1}), & x \ge ts_m \end{cases}$$

Hence $J_0^t v(x)$ has $m+1$ shocks $\chi_i(t) = ts_i$. □

Remark 2.68. We remark that, by the use of convex (resp. concave) envelope of H, it follows directly that, at each shock $\chi_i(t)$ of the viscosity solution $J_0^t v(x)$, with the jump of derivatives p_i, p_{i+1}, the graph of H between p_i and p_{i+1} lies above (resp. below) the segment joining $(p_i, H(p_i))$ and $(p_{i+1}, H(p_{i+1}))$. This is called the entropy condition for viscosity solutions, see Theorem 2.76 in the next section.

4. See the beginning of the section 2.5.2 for a precise definition of shock.

Now we are willing to investigate the minmax solution under the same hypotheses. We first give a profile for the wave front. Recall that the generating family is given by

$$S_t(x, x_0, y_0) = v(x_0) + xy_0 - tH(y_0) - x_0y_0$$

and the wave front at time t is

$$\mathcal{F}^t = \mathcal{F}^t(v) := \{(x, S_t(x, x_0, y_0)) | y_0 \in \partial v(x_0), x \in x_0 + t\partial H(y_0)\}$$

For any subset $A \subset \mathbb{R}$, define

$$\mathcal{F}^t_A := \mathcal{F}^t|_{\{x_0 \in A\}} = \{(x, S_t(x, x_0, y_0)) \in \mathcal{F}^t, x_0 \in A\}$$

We claim that the wave front $\mathcal{F}^t$ for $\varphi^t_H(\partial v)$ with v and H piecewise linear (with finite pieces) is formed by pieces of straight line segments. Indeed,

1) $\mathcal{F}^t_+ := \mathcal{F}^t_{\{x_0>0\}}$ and $\mathcal{F}^t_- := \mathcal{F}^t_{\{x_0<0\}}$ are two lines with slope p_+ and p_- respectively. Take the case $x_0 < 0$ for example, one has $y_0 = v'(x_0) = p_-$, and

$$\mathcal{F}^t_{x_0} = \{z(x_0, y) = (x_0 + ty, v(x_0) + t(yp_- - H(p_-))) : y \in \partial H(p_-)\}.$$

then for any x_0, $x_0' < 0$, y, $y' \in \partial H(p_-)$, the chord connecting $z(x_0, y)$ and $z(x_0', y')$ is of slope p_- .

2) Without loss of generality, we assume that $p_- = p_0' < \cdots < p_k' = p_+$ (*or* $p_+ = p_0' < \cdots < p_k' = p_-$)are the breakpoints of H between p_- and p_+. Then $\mathcal{F}^t_0 = \mathcal{F}^t|_{\{x_0=0\}}$ consists of k line segments with slope p_i' which corresponds to the breakpoint p_i'. This can be seen from the formula

$$\mathcal{F}^t_0 = \{z(y, p) = (ty, t(yp - H(p))) : p \in [p_+, p_-], y \in \partial H(p)\}.$$

If p_+, p_- are not breakpoints of H, then $\mathcal{F}^t_0$ loses two line segments of slopes p_+ and p_-, but the whole wave front $\mathcal{F}^t$ does not change in the presence of the two segments $\mathcal{F}^t_-$ and $\mathcal{F}^t_+$.

Lemma 2.69. *For the Riemann problem with piecewise linear Hamiltonian, if the graph of H between p_- and p_+ lies above (resp. below) the segment joining $(p_-, H(p_-))$ and $(p_+, H(p_+))$ assuming $p_- < p_+$ (resp. $p_- > p_+$), then $\mathcal{F}^t_0$ lies below (resp. above) the graph of the viscosity solution $J^t_0 v(x)$ in the wave front.*

Proof. We will prove the case where $p_- < p_+$, while the other is similar. Remark that the hypothesis is equivalent to saying that $(H|_{[p_-,p_+]})^{\smile}$ have no breakpoint in (p_-, p_+). For convenience, we may assume that $p_\pm$ are not breakpoints of H. The viscosity solution $J^t_0 v(x)$ has a shock $\chi(t)$ which $\mathcal{F}^t_-$ and $\mathcal{F}^t_+$ intersect,

$$\chi(t) = \frac{H(p_-) - H(p_+)}{p_- - p_+} t = x_-(t) + tH'(p_-) = x_+(t) + tH'(p_+)$$

for some $x_-(t) < 0$ and $x_+(t) > 0$. By the description of the formation of $\mathcal{F}^t$ before, it is sufficient to show that, at $x = \chi(t)$, $\mathcal{F}^t_0$ lies below $(\chi(t), p_-x_-(t) + t(p_-H'(p_-) - H(p_-)))$. The points in $\mathcal{F}^t_0$ are given by $(ty, t(yp - H(p)))$ with $y \in \partial H(p)$ and p breakpoint of H in (p_-, p_+). Let $ty = \chi(t) = x_-(t) + tH'(p_-)$,

$$\begin{aligned} & t(yp - H(p)) - (p_-x_-(t) + t(p_-H'(p_-) - H(p_-))) \\ = \; & t(yp - H(p)) - t(yp_-H(p_-)) \\ = \; & t(p - p_-)(y - \frac{H(p) - H(p_-)}{p - p_-}) \leq 0 \end{aligned}$$

where the inequality comes from $y = \frac{H(p_-)-H(p_+)}{p_- - p_+}$ and the hypothesis on the graph of H. Hence the $\mathcal{F}_0^t$ lies below the graph of $J_0^t v(x)$. In particular, they can intersect only at $\chi(t)$. $\square$

Example 2.70. The following two figures illustrate the wave front relate to the cases where $(H|_{[p_-,p_+]})^{\smile}$ has one breakpoint (Figure 2.1) and no breakpoint in (p_-, p_+) (Figure 2.2) respectively. The viscosity solutions are the maximum in the wave front.

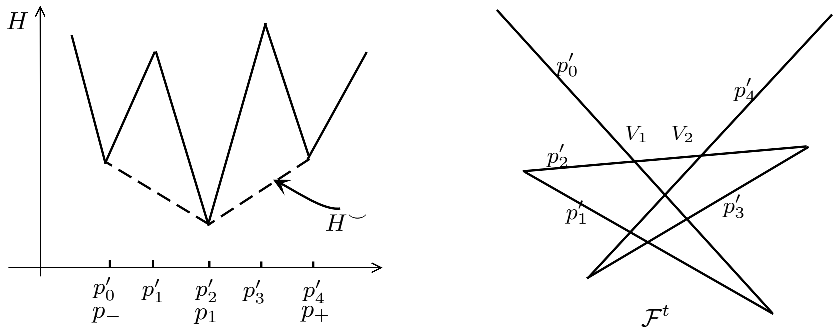


Figure 2.1: wave front

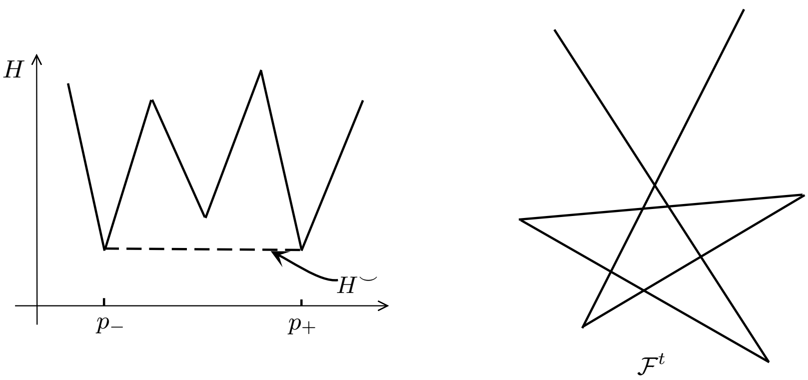


Figure 2.2: wave front

Definition 2.71. We say that $a \in \mathbb{R}$ admits a descending (resp.ascending) cycle if there is a descending (resp.ascending) cycle σ along which a is the maximum (resp.minimum) of the generating function S.

Lemma 2.72. *If $a \in \mathbb{R}$ admits at the same time a descending cycle and an ascending cycle, then a is both the minmax and maxmin value.*

Proof. By definition, it is easy to see that $\inf \max S \le a \le \sup \min S$. The inverse inequality follows from the fact that a descending cycle and an ascending cycle must intersect. $\square$

Proposition 2.73. *For the Riemann problem with piecewise linear Hamiltonian, the minmax solution $R_0^t v(x)$ coincides with the viscosity solution.*

Proof. We first remark that for an arbitrary initial function, the minmax and the viscosity solution may differ immediately.

Consider the Riemann problem with initial value $v(x) = p_-x$, $x \le 0$, $v(x) = p_+x$, $x > 0$, with $p_- < p_+$. It is sufficient to prove that $R_0^t v(x)$ is piecewise linear and has $m+1$ shocks $\chi_i(t) = ts_i$, $0 \le i \le m$, where $s_i = \frac{H(p_{i+1})-H(p_i)}{p_{i+1}-p_i}$, with $p_- = p_0 < \cdots < p_{m+1} = p_+$ the breakpoints of the convex envelope $(H|_{[p_-,p_+]})^{\smile}$.

For a fixed time t, let V_i, $0 \le i \le m$, be the intersection point of the line segments corresponding to p_i and p_{i+1} in the wave front $\mathcal{F}^t$

$$V_i = (ts_i, t(s_i p_i - H(p_i))) = (ts_i, t(s_i p_{i+1} - H(p_{i+1}))).$$

We want to show that the minmax $R_0^t v(x)$ is selected from $\mathcal{F}^T$ with the segments $\overline{V_i V_{i+1}}$, $0 \le i \le m-1$,

$$\overline{V_i V_{i+1}} = \{(ts, t(sp_{i+1} - H(p_{i+1}))) : s \,\text{between}\, s_i \,\text{and}\, s_{i+1}\}.$$

Fix any i, $0 \le i \le m-1$, denote $c_i^s := t(sp_{i+1} - H(p_{i+1}))$. We claim that (ts, c_i^s) admits a descending simplex, i.e. there exists a descending simplex σ_i^s such that

$$\max_{z_0 \in \sigma_i^s} S_t(ts, z_0) = c_i^s.$$

Write explicitly

$$\begin{aligned} S_t(ts, z_0) - c_i^s &= p_+x_0 - tH(y_0) + (ts - x_0)y_0 - t(sp_{i+1} - H(p_{i+1})) \\ &= x_0(p_+ - y_0) + tA_{i+1,s}(y_0), \quad \text{if } x_0 \ge 0. \\ S_t(ts, z_0) - c_i^s &= x_0(p_- - y_0) + tA_{i+1,s}(y_0), \quad \text{if } x_0 \le 0. \end{aligned}$$

where

$$A_{i+1,s}(y_0) := \begin{cases} (y_0 - p_{i+1})\left(s - \frac{H(y_0)-H(p_{i+1})}{y_0-p_{i+1}}\right), & \text{for} \quad y_0 \ne p_{i+1}, \\ 0, \quad \text{otherwise.} \end{cases}$$

By the convexity of $(H|_{[p_-,p_+]})^{\smile}$, we get

$$A_{i+1,s}(y_0) \le 0, \quad y_0 \in [p_-, p_+], \, s \,\text{between}\, s_i \,\text{and}\, s_{i+1}.$$

Note that the minmax and maxmin depend on the Hamiltonian H only on the compact set $[p_-, p_+]$. For each fixed i, we can modify H outside $[p_-, p_+]$ by requiring

$$\frac{H(y_0) - H(p_{i+1})}{y_0 - p_{i+1}} = \begin{cases} \frac{H(p_-)-H(p_{i+1})}{p_- - p_{i+1}}, & y_0 \le p_- \\ \frac{H(p_+)-H(p_{i+1})}{p_+ - p_{i+1}}, & y_0 \ge p_+ \end{cases}$$

so that

$$A_{i+1,s}(y_0) \le 0, \quad \forall y_0 \in \mathbb{R}$$

We construct a descending simplex $\sigma_i^s = (x_0(y_0), y_0)$ as follows:

$$x_0(y_0) = \begin{cases} \frac{tA_{i+1,s}(y_0)}{y_0 - p_-} + \theta(y_0), & y_0 \in [p_{i+1}, \infty), \\ \frac{tA_{i+1,s}(y_0)}{y_0 - p_+} - \theta(y_0), & y_0 \in (-\infty, p_{i+1}] \end{cases}$$

where $\theta : \mathbb{R} \to \mathbb{R}$ is a non negative continuous function, with

$$\theta(y_0) = \begin{cases} 0, & y_0 \in [p_-, p_+] \\ |y_0|, & |y_0| \geq C \end{cases}$$

with $C > 0$ a large constant. One sees that σ_i^s is a descending simplex since the first term in $x_0(y_0)$ is bounded, thus

$$\lim_{|y_0| \to \infty} |y_0|^{-1}((x_0(y_0), y_0) - (y_0, y_0)) = 0$$

Furthermore,

$$b_i^s(y_0) := S_t(ts, (x_0(y_0), y_0)) - c_i^s = 0, \quad \text{if } y_0 \in [p_-, p_+]$$

For $y_0 \in (p_+, \infty)$, the first term in $x_0(y_0)$ is negative, one can verify that $b_i^s(y_0)$ is negative both in the cases where $x_0(y_0) \leq 0$ and $x_0(y_0) \geq 0$. Similarly, we can show that $b_i^s(y_0) \leq 0$ for $y_0 \in (-\infty, p_-)$. Therefore c_i^s admits σ_i^s as a descending simplex.

On the other hand, replacing θ by $-\theta$ will give us an ascending simplex for c_i^s, i.e. $b_i^s(y_0) \geq 0$, with equality for $y_0 \in [p_-, p_+]$. Hence c_i^s is at the same time a minmax and maxmin value by Lemma 2.72.

For the case where $p_- > p_+$, that is, when v is concave, we should take the concave envelop $(H|_{[p_+, p_+]})^\frown$ and the proof is similar. □

Now suppose that v is piecewise linear continuous (with finite pieces). Thanks to its local nature [5] and semi-group property, one can construct the corresponding viscosity solution by viewing v as a combination of local Riemann initial data and chasing the interactions between the shocks of the corresponding local Riemann problems. This is the so-called *front tracking* method, which was first proposed by C.Dafermos ([30]).It consists in the following inductive procedure: every time there are collisions between the shocks, that is, the shocks of different local Riemann problems meet, we restart by considering the result state as a new initial function and propagate until the next time of collision. In the presence of finite propagating speed of characteristics, the number of shocks of the consequent solution will never blow up (ref. Lemma 2.6,[40]).

Proposition 2.74. *Assume that $v(x)$ is a convex piecewise linear continuous function with a finite number of discontinuities of $dv(x)$, and H is a piecewise linear continuous function with a finite number of breakpoints in $\{|p| \leq \|\partial v\|\}$. Then the minmax $R_0^t v(x)$ coincides with the viscosity solution.*

Proof. It is enough prove that for any $s > 0$ small enough, $R_s^t \circ R_0^s v(x)$ is the viscosity solution for any $t > s$, then by Lemma 2.60, $R_0^t v(x)$ is the viscosity solution. Let $s > 0$ be small enough such that $R_0^s v(x) = J_0^s v(x)$. This is possible since for $s > 0$ small, we can get $R_0^s v(x)$ and $J_0^s v(x)$ as a combination of local Riemann problems [6] where they are equal by Proposition 2.73. Denote by $v_s = R_0^s v$, $G(t) = \mathrm{Gr}(J_0^t v)$ the graph of $J_0^t v$ and $\mathring{\mathcal{F}}(t) = \mathcal{F}^{t-s}(v_s) \setminus G(t)$. According to lemma 2.65,

$$J_0^t v(x) = J_s^t v_s(x) \geq \max\{u : (x, u) \in \mathcal{F}^{t-s}(v_s)\},$$

5. By the definition of the viscosity solution, if $\{U_i\}_i$ is an open cover of $(0, +\infty) \times \mathbb{R}$, and every restriction $u|_{U_i}$ is a viscosity solution, then u is a viscosity solution. In particular, for a piecewise C^1 function u to be viscosity, it is enough that its shocks, that is, the places where it is not differentiable, verify the condition to be viscosity.

6. The local property of the minmax means that the value $R_0^t v(x)$ depends on the restriction of v to a neighbourhood of the set $\{x_0 : |x - x_0| \leq t\|\partial H\|\}$.

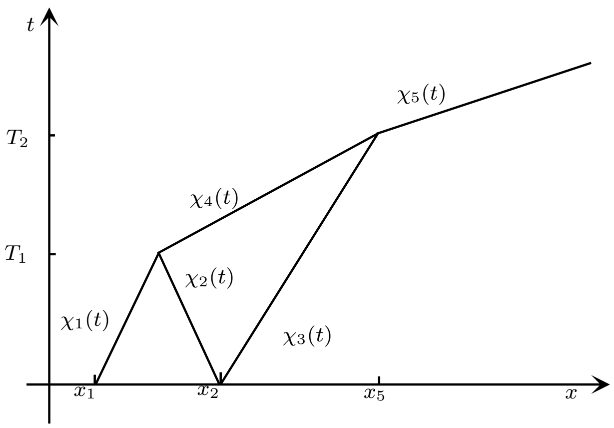


Figure 2.3: Front tracking (collision of shocks)

that is $\mathring{\mathcal{F}}(t)$ lies below $G(t)$ for all $t > s$. We claim that, for $t > s$, $G(t) \subset \mathcal{F}^{t-s}(v_s)$ and $\mathrm{cl}(\mathring{\mathcal{F}}(t)) \cap G(t) \subset I(t)$ where cl denotes the closure of a set, and $I(t) := \{X_i(t) := (\chi_i(t), J_0^t(\chi_i(t)))\}$ with i runs through all the discontinuities of the derivatives of $J_0^t v$ at time t.

Let x_i, $1 \le i \le n$, be the breakpoints of v_s; by the hypothesis of convexity of v, the $p_i^\pm := v_s'(x_i\pm)$ satisfy $p_i^- \le p_i^+ = p_{i+1}^- < p_{i+1}^+$. Let $u^i(s,t)(x)$ be the viscosity solution of the local Riemann problem with initial value $v^i(x) = v_s(x)$, $x_{i-1} \le x \le x_{i+1}$. By Remark 2.68, $(H|_{[p_i^-, p_i^+]})^{\smile}$ has no breakpoints in (p_i^-, p_i^+), thus each u^i has a shock $\chi_i(t)$, and consist of two lines with slope $p_i^\pm$ for $t > s$. For $t \in (0, T_1)$ where T_1 is the first time there are collisions of shocks, $J_0^t v(x) = J_s^t v_s(x)$ is given by $u^i(s,t)(x)$ for $\chi_{i-1}(t) \le x \le \chi_{i+1}(t)$. Hence $G(t) \subset \mathcal{F}^{t-s}(v_s)$ for $t \in (s, T_1]$.

If $\chi_j(t)$ and $\chi_{j+1}(t)$ be two adjacent shocks colliding at T_1, that is,

$$\chi_j(s) < \chi_{j+1}(s),\ \dot\chi_j(t) > \dot\chi_{j+1}(t),\ \text{and}\ \chi_j(t) < \chi_{j+1}(t)\ \text{for}\ t \in (s, T_1),\ \chi_j(T_1) = \chi_{j+1}(T_1).$$

Since

$$\dot\chi_{j+1}(t) = \frac{H(p_{j+1}^+) - H(p_{j+1}^-)}{p_{j+1}^+ - p_{j+1}^-} < \dot\chi_j = \frac{H(p_j^+) - H(p_j^-)}{p_j^+ - p_j^-}$$

and $(H|_{[p_-^k, p_+^k]})^{\smile}$ has no breakpoints in (p_-^k, p_+^k), $k = j, j+1$, we get that $(H|_{[p_-^j, p_+^{j+1}]})^{\smile}$ has no breakpoints in (p_-^j, p_+^{j+1}). Hence for $t \in (T_1, T_2)$, $x \in [\chi_{j-1}(t), \chi_{j+1}(t)]$, where T_2 is second time of collisions of shocks of $J_0^t v(x)$, $J_0^t v(x)$ is given by the left branch of $u^j(s,t)$ and the right branch of $u^{j+1}(s,t)$, whose intersection generates a new shock $\tilde\chi(t)$, while the original shocks χ_j and χ_{j+1} disappear (see figure 2.4). Thus $G(t) \subset \mathcal{F}^{t-s}(v_s)$ for $t \in (T_1, T_2]$. The cases where there are more than two shocks collide at the same time can be argued similarly. In this way, we can prove that, for $t > s$, $G(t) \subset \mathcal{F}^{t-s}(v_s)$. We remark that the number of elements in $I(t)$ decrease whenever there is collision of shocks. The claim that $\mathrm{cl}(\mathring{\mathcal{F}}(t)) \cap G(t) \subset I(t)$ follows from the fact that every $\mathcal{F}^{t-s}_{\chi_i(s)}(v_s)$ lies below the graph of $u^k(s,t)$ with possible intersection only at $\chi_i(t)$ and the line segments in $\mathcal{F}^{t-s}_{\chi_i(s)}(v_s)$ has slopes between $[p_i^-, p_i^+]$ thus can not intersect the graph of $u^k(s,t)$ for $k \ne i$ by the convexity of v_s.

Let $t_0 = \max\{t : R_s^t v_s(x) = J_0^t(x), \forall x\}$. Using again the local property of the minmax and viscosity solution, there exists some $\epsilon > 0$ such that $t_0 > s + \epsilon$. If $t_0 < +\infty$, then $R_s^{t_0+} v_s = \lim_{t \downarrow t_0} R_s^t v_s$ contains a curve of positive length in $\mathrm{cl}(\mathring{\mathcal{F}}(t_0))$ connecting some

$X_i(t_0)$ and $X_{i'}(t_0)$. Indeed, since $\mathrm{cl}(\mathring{\mathcal{F}}(t))$ contains a finite number of line segments whose slopes are fixed for all t, a curve connecting $X_i(t)$ and $X_i'(t)$ can not deform continuously to the line in $G(t_0)$ connecting $X_i(t_0)$ and $X_i'(t_0)$ as $t \downarrow t_0$. Therefore $R_s^{t_0} v_s \neq R_s^{t_0+} v_s$, which contradicts the continuity of the minmax with respect to time. Thus we complete the proof that $R_s^t v_s(x) = J_0^t v(x)$ for any $s > 0$ small enough.

□

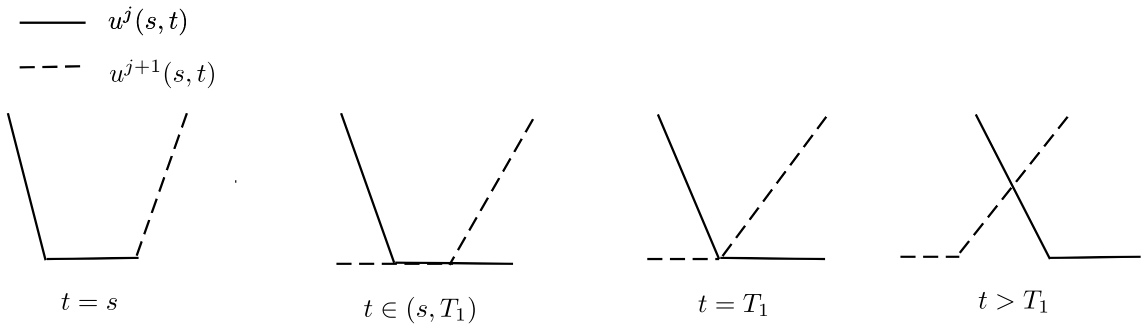


Figure 2.4

Theorem 2.75. *If v is convex or concave and globally Lipschitz and H continuous, then the minmax solution $R_0^t v(x)$ coincides with the viscosity solution of the (CL) equation .*

Proof. For any compact subset $K \subset \mathbb{R}$, fixed once for all, we can modify H and v so that H vanishes outside a neighborhood of $\mathcal{K} := \{|p| \leq \|\partial v\|\}$ and v is linear outside $\tilde{K} = \{|x_0 - x| \leq T\|\partial H\|, \forall x \in K\}$. For $k \in \mathbb{N}$, take piecewise linear interpolations v^k and H^k for v and H such that

$$|v - v^k|_{C^0} \leq 1/k, \quad |H - H^k|_{C^0} \leq 1/k$$

From Proposition 2.74, for $x \in K$, we have $R_{H_k}^t v_k(x) = J_{H_k}^t v_k(x)$. Hence by the continuity property of the minmax and viscosity solution, we get that $R_H^t v(x) = J_H^t v(x)$.

□

2.5.2 Singularities of viscosity solution

In this section, we will investigate the difference between the minmax and the viscosity solution for nonconvex Hamiltonians and initial functions. We shall see how the limit of minmax serves to describe the singularities of the viscosity solution.

Assume that $\mathcal{O} \subset (0, \infty) \times \mathbb{R}$ is an open set, $u \in C(\mathcal{O})$, and that there is a C^1 curve $t \to x = \chi(t)$, $t \in (t_1, t_2)$ dividing $\mathcal{O}$ into two open subsets $\mathcal{O}^+$ and $\mathcal{O}^-$, $\mathcal{O} = \mathcal{O}^+ \cup \chi \cup \mathcal{O}^-$. If $u \in C(\mathcal{O})$ and $u = u^+$ in $\mathcal{O}^+ \cup \chi$, $u = u^-$ in $\mathcal{O}^- \cup \chi$, where $u^\pm \in C^1(\mathcal{O}^\pm \cup \chi)$, $u_-|_\chi \neq u_+|_\chi$, we say that χ is a *shock* curve of u in the neighborhood $\mathcal{O}$.

The viscosity solution of the (CL) equation in one space variable is characterized equivalently by Oleinik's entropy condition:

Theorem 2.76 (Entropy condition,[46])**.** *Let $u \in C(\mathcal{O})$ and $u = u^+$ in $\mathcal{O}^+ \cup \chi$, $u = u^-$ in $\mathcal{O}^- \cup \chi$, where $u^\pm \in C^1(\mathcal{O}^\pm \cup \chi)$. Then u is the viscosity solution of the equation (CL) in $\mathcal{O}$ if and only if:*

1. *$u^\pm$ are classical solutions in $\mathcal{O}^\pm$ respectively,*

2. *The graph of H lies below (resp. above) the line segment joining the points $(p_t^-, H(p_t^-))$ and $(p_t^+, H(p_t^+))$ if $p_t^+ < p_t^-$ (resp. $p_t^- < p_t^+$), where $p_t^\pm := \partial_x u^\pm(t, \chi(t))$.*

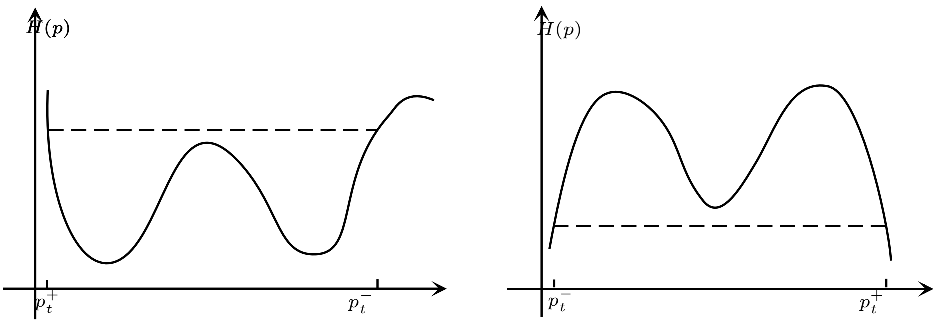


Figure 2.5: Entropy condition

In particular, we say that $\chi(t)$ *strictly* verifies the entropy condition if it verifies the entropy condition and the line segment joining $(p_t^-, H(p_t^-))$ and $(p_t^+, H(p_t^+))$ is not tangent to the graph of H.

Remark 2.77. 1. The entropy condition is a local nonlinear version of the linear Riemann problem given in the above section, where we choose the convex or concave envelop of H, depending on the convexity of the initial function, to construct the viscosity solution.

2. Equivalently, the entropy condition can be described, for example, when $p_t^+ < p_t^-$, as

$$\frac{H(w) - H(p_t^-)}{w - p_t^-} \geq \frac{H(p_t^+) - H(p_t^-)}{p_t^+ - p_t^-} (= \dot{\chi}(t)), \tag{2.19}$$

$$\text{or} \quad \frac{H(w) - H(p_t^+)}{w - p_t^+} \leq \frac{H(p_t^+) - H(p_t^-)}{p_t^+ - p_t^-} \tag{2.20}$$

for any w between p_t^+ and p_t^-, The "=" between brackets refers to the Rankine-Hugoniot condition which is a necessary condition for u to be a weak solution in distribution sense.

The formation of singularities of the viscosity solution of the (CL) equation are well studied, for example, in [46, 43]. If H and v are smooth, before a critical time $t_0 > 0$, the wave fronts are single-valued, and a classical smooth solution exists, given by the wave fronts. Beyond t_0, the viscosity solution has discontinuities in the derivatives. One may assume that only finitely many shocks of the viscosity solution are generated under some generic conditions on v and H, for example H having finitely many critical points and inflection points. Then, in the spirit of front tracking method, one can continue the viscosity solution further by constructing at subsequent time the solutions of all possible local Riemann problems:

$$\begin{cases} \partial_t u + H(\partial_x u) = 0, & (t,x) \in (\bar{t}, \bar{t} + \epsilon) \times (\bar{x} - \delta, \bar{x} + \delta), \\ u(\bar{t}, x) = \bar{v}(x). \end{cases} \tag{2.21}$$

where $\bar{v} \in C^\infty((\bar{x}-\delta, \bar{x}+\delta)\setminus\{\bar{x}\}) \cap C((\bar{x}-\delta, \bar{x}+\delta))$ for $\epsilon, \delta > 0$ sufficiently small. Here $\bar{v}'$ is assumed to be discontinuous only at the point $\bar{x}$ across which the viscosity criterion is satisfied; one also assume that $\mathcal{F}^t_+ := \mathcal{F}^t_{\{x_0>\bar{x}\}}(\bar{v})$ and $\mathcal{F}^t_- := \mathcal{F}^t_{\{x_0<\bar{x}\}}(\bar{v})$ are single-valued. Denote $u^\pm(t,\cdot)$ the functions whose graphs define $\mathcal{F}^t_\pm$.

Following G.T.Kossioris in [46], the shock curves of the viscosity solutions are either *genuine shocks* or *contact discontinuities.* The genuine shock curves are determined by the intersection of $\mathcal{F}^t_+$ and $\mathcal{F}^t_-$. Due to the presence of inflection points of H, $\mathcal{F}^t_\pm$ may have no common point. In that case, the viscosity solution is constructed by jointing u^+ and u^- with a *rarefaction-wave* type solution. There are *rarefaction waves* which are carried by the characteristic lines originating tangentially from the contact discontinuity and not the initial axis. See Figure 2.6.

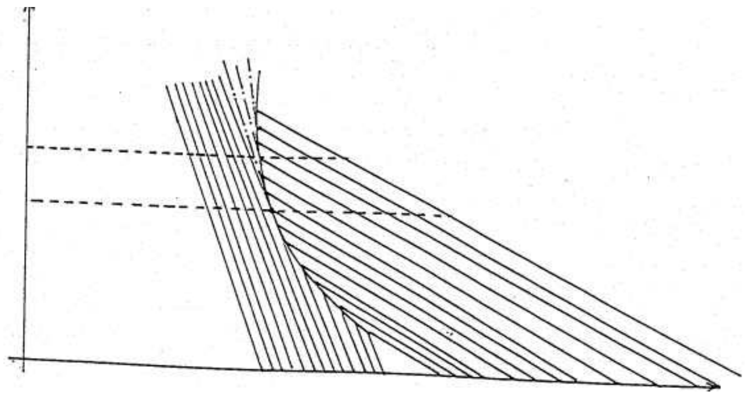


Figure 2.6

Now let us investigate the relation of the minmax solution and the viscosity solution for the local Riemann problem (2.21). If the shocks of the minmax $R^t_0\bar{v}(x)$ satisfy the entropy condition, then by Theorem 2.76, it is the viscosity solution. We will give a criterion for this to happen. It is a local nonlinear version of Lemma 2.69.

If $\mathcal{F}^t_+$ and $\mathcal{F}^t_-$ intersects, suppose that $\chi(t) = u^\pm(t, x^\pm_t)$, where $x^+_t > \bar{x}$ and $x^-_t < \bar{x}$ for $t > 0$ and denote $p^\pm_t := \partial_x u^\pm(t, \chi(t))$. We say that the C^1 curve $\chi(t)$ (strictly) verifies the entropy condition if (2) in Theorem 2.76 (strictly) holds.

Proposition 2.78. *For the local Riemann problem* (2.21)*, if the curve $\chi(t)$ given by the intersection of $\mathcal{F}^t_\pm$ strictly verifies the entropy condition, then $R^t_0\bar{v}(x)$ coincides with the viscosity solution.*

Proof. It follows directly from Theorem 2.76 that the viscosity solution u is composed by $u^\pm$ and that $\chi(t)$ is a shock of u. We assume that $p^+_0 < p^-_0$, the other case being similar. The wave front is $\mathcal{F}^t = \mathcal{F}^t_\pm \cup \mathcal{F}^t_{\bar{x}}$, where :

$$\mathcal{F}^t_{\bar{x}} = \{z(p) = (\bar{x} + tH'(p), \bar{v}(\bar{x}) + t(pH'(p) - H(p)))\ ,\ p \in [p^+_0, p^-_0]\}. \tag{2.22}$$

We claim that $\mathcal{F}^t_{\bar{x}}$ does not interfere in the selection of the minmax solution $R^t_0\bar{v}(x)$ for $(t,x) \in \mathcal{O} := (0,\epsilon)\times(\bar{x}-\delta, \bar{x}+\delta)$, $\epsilon, \delta > 0$ sufficiently small. For this purpose, it is enough to show that there is no point in $\mathcal{F}^t_{\bar{x}}$ lying below the shock $(\chi(t), u(t,\chi(t)))$, where

$$\begin{aligned} \chi(t) &= x^\pm_t + tH'(p^\pm_t), \\ u(t,\chi(t)) &= u^\pm(0, x^\pm_t) + t(p^\pm_t H'(p^\pm_t) - H(p^\pm_t)) \end{aligned}$$

Suppose $t \mapsto \alpha_t \in [p_0^+, p_0^-]$ is a C^1 function of t such that

$$z(\alpha_t) = (\bar{x} + tH'(\alpha_t), v(\bar{x}) + t(\alpha_t H'(\alpha_t) - H(\alpha_t))) \in \mathcal{F}^t_{\bar{x}} \cap \{x = \chi(t)\},$$

Define

$$A(t) = (u^-(0, x_t^-) + t(p_t^- H'(p_t^-) - H(p_t^-))) - (u(0, \bar{x}) + t(\alpha_t H'(\alpha_t) - H(\alpha_t))).$$

then $A(0) = 0$. Note that $\partial_x u^-(0, x_t^-) = \partial_x u^-(t, \chi(t))$ because they lie in the same characteristics,

$$\begin{aligned}
\dot{A}(t) &= p_t^- \dot{x}_t^- + p_t^- H'(p_t^-) - H(p_t^-) - \alpha_t H'(\alpha_t) + H(\alpha_t) + t(p_t^- H''(p_t^-)\dot{p}_t^- - \alpha_t H''(\alpha_t)\dot{\alpha}_t) \\
&= p_t^-(\dot{\chi}(t) - H'(p_t^-) - tH''(p_t^-)\dot{p}_t^-) + p_t^- H'(p_t^-) - H(p_t^-) - \alpha_t H'(\alpha_t) + H(\alpha_t) \\
&\quad + t(p_t^- H''(p_t^-)\dot{p}_t^- + \alpha_t(H'(\alpha_t) - \dot{\chi}(t))) \\
&= (p_t^- - \alpha_t)(\dot{\chi}(t) - \frac{H(\alpha_t) - H(p_t^-)}{\alpha_t - p_t^-}).
\end{aligned}$$

If $\chi(t)$ strictly verifies the entropy condition, then we have $\alpha_0 := \lim_{t>0, t\to 0} \alpha_t < p_0^-$. Indeed, if $\alpha_0 = p_0^-$, then

$$\dot{\chi}(0) := \lim_{t>0, t\to 0} \dot{\chi}(t) = \lim_{t>0, t\to 0} H'(\alpha_t) + tH''(\alpha_t)\dot{\alpha}_t = H'(\alpha_0) = H'(p_0^-)$$

which means that the shock χ does not strictly verify the entropy condition. Therefore, for sufficiently small time t, we can assume that $\alpha_t < p_t^-$, then by the entropy condition (2.19), we get $\dot{A}(t) < 0$, hence $A(t) < 0$. Now the only way to select a continuous section in $\mathcal{F}^t$ is choosing the two branches $u^\pm$ with the shock $\chi(t)$. Since the graph of the minmax solution $\{(x, R_0^t \bar{v}(x))\}$ is a continuous section in $\mathcal{F}^t$, we get

$$R_0^t \bar{v}(x) = \begin{cases} u^+(t, x), & (t, x) \in \mathcal{O}^+ \cup \chi \\ u^-(t, x), & (t, x) \in \mathcal{O}^- \cup \chi \end{cases}$$

□

Remark 2.79. With generic assumptions on H and v, the hypothesis of Proposition 2.78 is true for a small time after the first critical time t_0, until another critical time $t_\alpha > t_0$, see [46, 43].

Example 2.80. If H is convex (resp. concave), then in the wave front $\mathcal{F}^t$, $\mathcal{F}^t_{\bar{x}}$ always lie above (resp. below) $\mathcal{F}^t_\pm$, hence the min (resp. max) solution is the viscosity solution. See Figure 2.7.

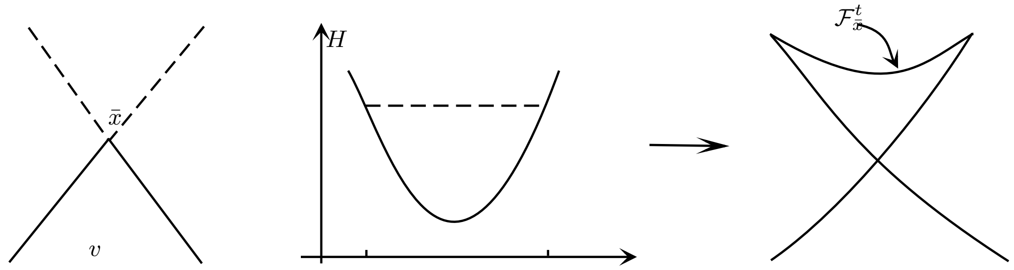


Figure 2.7

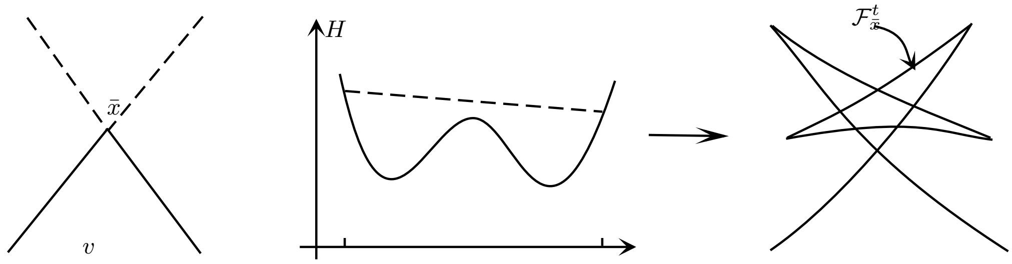


Figure 2.8

Example 2.81. If H is non convex, but the intersection $\chi(t)$ of $\mathcal{F}^t_{\pm}$ strictly satisfies the entropy condition, the configuration of the wave front are depicted in Figure 2.8.

Now we turn to the case that the minmax has a shock $\chi(t)$ which do not satisfy the entropy condition. We will use the Theorem 2.53 to show that, the viscosity solution will have contact type shock and the mysterious rarefaction waves are emitting from the singularities of the minmax iterated step by step through the limiting process.

Let $\chi(t)$ be a C^1 shock of $R^t_0\bar{v}(x)$ of the local Riemann problem (2.21), and $\chi(t)$ violates the entropy condition. In this case, $\chi(t)$ can not be the intersection of $\mathcal{F}^t_{\pm}(\bar{v})$, otherwise the minmax $R^t_0\bar{v}(x)$ will form a semi-group with respect to t, and by Proposition 2.44, it is viscosity.

Given a subdivision $\{\zeta_n\}$ of $(0,\epsilon)$, where $\zeta_n = \{0 = t_0 < t_1^{(n)} < \cdots < t_n^{(n)} = \epsilon\}$, let $\chi_n(t)$ be a continuous, piecewise C^1 shock of the related iterated n-step minmax $u_n(t,x) := R^t_{0,n}\bar{v}(x)$. Denote $p_t^{\pm} := \partial_x u_n(t, \chi_n(t)\pm)$ and $u_n^k(x) = u_n(t_k^{(n)}, x)$. By Lemma 2.60, for $|\zeta_n|$ small enough, there is violation of the entropy condition for the iterated minmax: for $t \in (t_k^{(n)}, t_{k+1}^{(n)}]$, $\chi_n(t)$ are constructed by the intersection of two branches of $\mathcal{F}^{t-t_k^{(n)}}(u_n^k)$, one of which comes from $\mathcal{F}^{t-t_k^{(n)}}_{\chi_n(t_k^{(n)})}(u_n^k)$. Assuming that this branch lies in the left of $\chi_n(t)$, then

$$\chi_n(t) = \chi_n(t_k^{(n)}) + (t - t_k^{(n)})H'(p_t^-), \quad t \in (t_k^{(n)}, t_{k+1}^{(n)}]$$

where p_t lies between $p^+_{t_k^{(n)}}$ and $p^-_{t_k^{(n)}}$ for $t \in (t_k^{(n)}, t_{k+1}^{(n)}]$. Note that $\chi_n(t)$ satisfies the Rankine-Hugoniot condition, hence

$$\dot{\chi}_n(t_k^{(n)}) = \frac{H(p^+_{t_k^{(n)}}) - H(p^-_{t_k^{(n)}})}{p^+_{t_k^{(n)}} - p^-_{t_k^{(n)}}} = H'(p^-_{t_k^{(n)}}),\ 0 \le k \le n-1.$$

Thus the limit $\bar{R}^t_0\bar{v}(x)$ of the sequence of iterated minmax $\{R^t_{0,n}\bar{v}(x)\}_n$ has a C^1 shock $\bar{\chi}(t)$ satisfying

$$\dot{\bar{\chi}}(t) = \frac{H(p_t^+) - H(p_t^-)}{p_t^+ - p_t^-} = H'(p_t^-), \quad 0 \le t \le \epsilon. \tag{2.23}$$

By Theorem 2.53, the limit $\bar{R}^t_0\bar{v}(x)$ is the viscosity solution, hence the limiting iterated minmax procedure explains how to form a contact type shock and what are the rarefaction waves.

Different from the case in Proposition 2.78, if the singularity of $\bar{v}$ satisfied the entropy condition, but not strictly, that is, denoting $p_0^{\pm} := \bar{v}'(\bar{x}\pm)$, the line joining $(p_0^+, H(p_0^+))$ and

$(p_0^-, H(p_0^-))$ is tangent to the graph of H, then the construction of viscosity solution of the local Riemann problem falls into different subcases depend on the analytic information such as the sign of $\bar{v}''(x)$, $x \neq \bar{x}$, see [46, 43]. We will not perform here the technical details but illustrate the formation of a contact shock in a concrete example.

First, we have some key rules to characterize $\mathcal{F}^t_{\bar{x}}$ ((2.22)):

1. The cusps in $\mathcal{F}^t_{\bar{x}}$ correspond to the inflection points of H; the number of branches in $\mathcal{F}^t_{\bar{x}}$ [7] is the number of inflection points of H and the convexity of each branch of $\mathcal{F}^t_{\bar{x}}$ between two cusps coincide with that of H between two inflection points;

2. The slope of $\mathcal{F}^t_{\bar{x}}$ at $z(p)$ is p;

3. $\mathcal{F}^t_{\bar{x}}$ and the two genuine branches $u^\pm$ are joined in a C^1 smooth manner at the extremities $z(p_0^\pm)$ with slope $p_0^\pm$.

Example 2.82 (Rarefaction)**.** Consider

$$v(x) = \begin{cases} -x(x-1), & x \leq 0 \\ x(x-1), & x \geq 0 \end{cases}, \quad H(p) = -p^3 + p^2 + p$$

Look at a neighborhood of the singularity $x = 0$ of v.

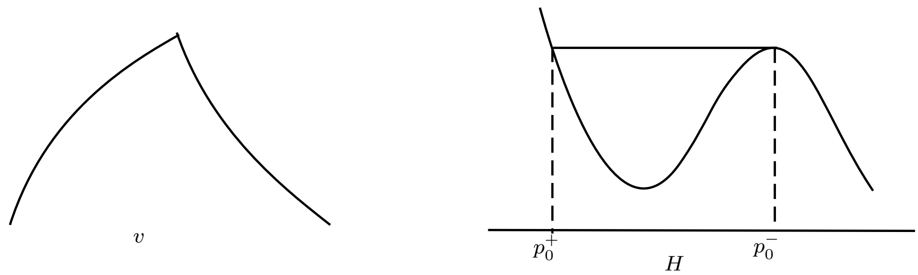


Figure 2.9

For $t > s > 0$ small, the geometric solution and wave fronts are depicted as follows in Figure 2.10 and Figure 2.11.

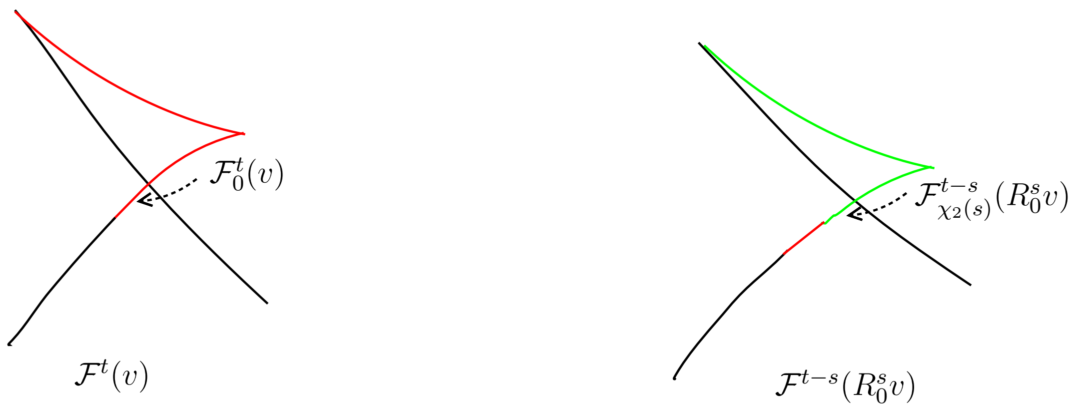


Figure 2.10

7. A branch is a partially defined C^1 curve.

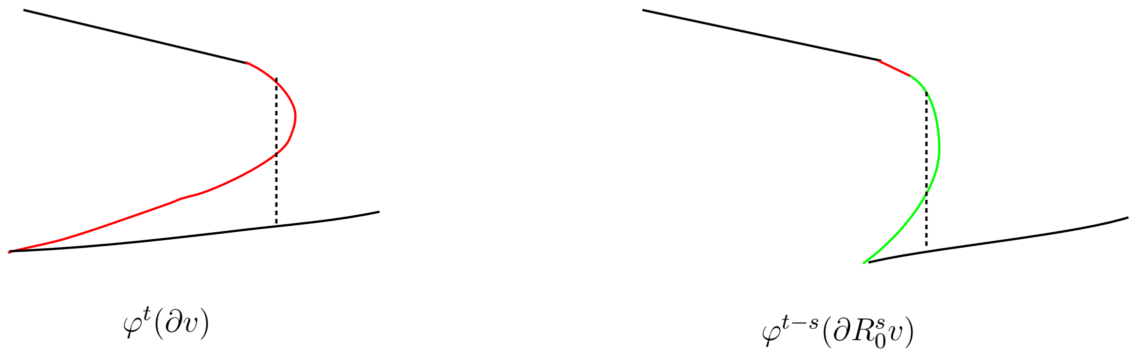


Figure 2.11

Figure 2.12

In the wave front $\mathcal{F}^t(v)$, the two branches in blue are genuine branches, and the curve in red is $\mathcal{F}_0^t(v)$. The 1-step minmax $R_0^t v(x)$ takes the minimum in the wave front, and the shock $\chi_1(t)$ of $R_0^t v(x)$ is given by the intersection of $\mathcal{F}_+^t$ and a branch of $\mathcal{F}_0^t(v)$. The 2-step minmax $R_s^t \circ R_0^s v(x)$ has a shock $\chi_2(t)$ which, for $t > s$, is given by the intersection of $\mathcal{F}_+^t$ and a branch of $\mathcal{F}_{\chi_2(s)}^{t-s}(R_0^s v)$. By the limiting process of iterated minmax, we then get a contact shock as explained above.

The viscosity solution is thus not contained in the geometric solution $\mathcal{F}^t(v)$. See Figure 2.12.

We remark that, in general, for a local Riemann problem (2.21) where the entropy condition is not required to be satisfied for the singularity of initial function $\bar{v}$, even when the two branches $\mathcal{F}_\pm^t(\bar{v})$ separate, the minmax $R_0^t \bar{v}(x)$ can still be the viscosity solution. This is the case when $\bar{v}$ is convex (e.g. Figure 2.1). A proper notion of the "genuine" shock of the viscosity solution may be given by the intersection of two branches in $\mathcal{F}^t(\bar{v}) = \mathcal{F}_\pm^t \cup \mathcal{F}_{\bar{x}}^t$, where "genuine" means that it is generated by two incoming characteristic from the initial axis. Summarizing, the type of the shocks of the viscosity solution is related to the semi group property of the minmax: for the local Riemann problem, if the minmax forms a semi-group, then it is the viscosity solution whence has genuine shocks; otherwise, there are rarefaction waves born from the limiting process of iterated minmax, which form a contact shock.

Chapter 3

Subtleties of the minmax selector

In the previous chapters, we have used the minmax (maxmin) as a graph selector to solve the Hamilton-Jacobi equations. In this chapter, we will look at the minmax-maxmin for its own sake, asking the following natural questions:

1. Is there only one "minmax"? More precisely, does the minmax depend on the coefficients used to define it?

2. Are the minmax and maxmin the same?

3.1 Introduction

The minmax has been defined using homology or cohomology with various coefficient rings, for example $\mathbb{Z}$ in [21, 62], $\mathbb{Q}$ in [14] and $\mathbb{Z}_2$ in [53]. Also, in [62, 64], the maxmin was mentioned as a natural analogue to the minmax. But there is no evidence showing that all these critical values coincide. G. Capitanio has given a proof [14] that the maxmin and minmax for homology with coefficients in $\mathbb{Q}$ are equal, but the criterion he uses (Proposition 2 in [14]) is not correct—see Remark 3.25 hereafter.

We will investigate the maxmin and minmax for a general function quadratic at infinity, not necessarily related to Hamilton-Jacobi equations. We give both algebraic and geometric proofs that the minmax and maxmin with coefficients in a field coincide; the geometric proof, based on Barannikov's Jordan normal form for the boundary operator of the Morse complex, improves our understanding of the problem.

A counterexample for coefficients in $\mathbb{Z}$, due to F. Laudenbach [37], is constructed using Morse homology; in this example, moreover, the minmax-maxmin for coefficients in $\mathbb{Z}_2$ is not the same as for coefficients in $\mathbb{Q}$. However, if the minmax and maxmin for coefficients in $\mathbb{Z}$ coincide, then all three minmax-maxmin critical values are equal.

3.2 Maxmin and Minmax

Hypotheses and notation

We denote by X the vector space $\mathbb{R}^n$ and by f a real function on X, *quadratic at infinity* in the sense that it is continuous and there exists a nondegenerate quadratic form $Q : X \to \mathbb{R}$ such that f coincides with Q outside a compact subset.

Let $f^c := \{x|f(x) \leq c\}$ denote the sub-level sets of f. Note that for c large enough, the homotopy types of f^c, f^{-c} do not depend on c, we may denote them as f^∞ and

$f^{-\infty}$. Suppose the quadratic form Q has Morse index λ, then the homology groups with coefficient ring R are

$$H_*(f^\infty, f^{-\infty}; R) \simeq \begin{cases} R & \text{in dimension}\,\lambda \\ 0 & \text{otherwise} \end{cases}$$

Consider the homomorphism of homology groups

$$i_{c*} : H_*(f^c, f^{-\infty}; R) \to H_*(f^\infty, f^{-\infty}; R)$$

induced by the inclusion $i_c : (f^c, f^{-\infty}) \hookrightarrow (f^\infty, f^{-\infty})$.

Definition 3.1. If Ξ is a generator of $H_\lambda(f^\infty, f^{-\infty}; R)$, we let

$$\underline{\gamma}(f, R) := \inf\{c : \Xi \in \operatorname{im}(i_{c*})\},$$

i.e. $\underline{\gamma}(f, R) = \inf\{c : i_{c*}H_\lambda(f^c, f^{-\infty}; R) = H_\lambda(f^\infty, f^{-\infty}; R)\}$.

Similarly, we can consider the homology group

$$H_*(X \setminus f^{-\infty}, X \setminus f^\infty; R) \simeq \begin{cases} R, & \text{in dimension}\, n-\lambda \\ 0, & \text{otherwise} \end{cases}$$

and the homomorphism

$$j_{c*} : H_*(X \setminus f^c, X \setminus f^\infty; R) \to H_*(X \setminus f^{-\infty}, X \setminus f^\infty; R)$$

induced by $j_c : (X \setminus f^c, X \setminus f^\infty) \hookrightarrow (X \setminus f^{-\infty}, X \setminus f^\infty)$.

Definition 3.2. If Δ is a generator of $H_{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^\infty; R)$, we let

$$\begin{aligned} \overline{\gamma}(f, R) &:= \sup\{c : \Delta \in \operatorname{im}(j_{c*})\} \\ &= \sup\{c : j_{c*}H_{n-\lambda}(X \setminus f^c, X \setminus f^\infty; R) = H_{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^\infty; R)\}. \end{aligned}$$

Lemma 3.3. *One has that*

$$\begin{aligned} \underline{\gamma}(f, R) &= \inf\max f := \inf_{[\sigma]=\Xi}\max_{x\in|\sigma|} f(x) \\ \overline{\gamma}(f, R) &= \sup\min f := \sup_{[\sigma]=\Delta}\min_{x\in|\sigma|} f(x), \end{aligned}$$

where σ is a relative cycle and $|\sigma|$ denotes its support. We call σ a descending (resp. ascending) cycle if $[\sigma] = \Xi$ (resp. $[\sigma] = \Delta$).

Proof. A descending cycle σ defines a homology class in $H_\lambda(f^c, f^{-\infty}; R)$ if and only if $|\sigma| \subset f^c$, in which case one has $\max\limits_{x\in|\sigma|} f(x) \le c$, hence $\underline{\gamma}(f, R) \ge \inf\max f$; choosing $c = \max\limits_{x\in|\sigma|} f(x)$, we get equality. The case of $\overline{\gamma}$ is identical. □

Definition 3.4. $\underline{\gamma}(f, R)$ is called a minmax of f and $\overline{\gamma}(f, R)$, a maxmin .

Remark 3.5. As we shall see later, in view of Morse homology, the names are proper generically for Morse-excellent functions.

One can also consider cohomology instead of homology and define

$$\begin{aligned} \underline{\alpha}(f, R) &:= \inf\{c : i_c^* \neq 0\}, \quad i_c^* : H^\lambda(f^\infty, f^{-\infty}; R) \to H^\lambda(f^c, f^{-\infty}; R) \\ \overline{\alpha}(f, R) &:= \sup\{c : j_c^* \neq 0\}, \quad j_c^* : H^{n-\lambda}(X \setminus f^{-\infty}, X \setminus f^\infty; R) \to H^{n-\lambda}(X \setminus f^c, X \setminus f^\infty; R). \end{aligned}$$

Proposition 3.6 ([64], Proposition 2.4)**.** *When X is R-oriented,*

$$\overline{\alpha}(f,R)=\underline{\gamma}(f,R) \quad \text{and} \quad \underline{\alpha}(f,R)=\overline{\gamma}(f,R).$$

Proof. We establish for example the first identity: one has the commutative diagram

$$\begin{array}{ccc} H_\lambda(f^c,f^{-\infty};R) & \simeq & H^{n-\lambda}(X\setminus f^{-\infty},X\setminus f^c;R) \\ \downarrow^{i_{c*}} & & \downarrow \\ H_\lambda(f^\infty,f^{-\infty};R) & \simeq & H^{n-\lambda}(X\setminus f^{-\infty},X\setminus f^\infty;R) \\ \downarrow & & \downarrow^{j_c^*} \\ H_\lambda(f^\infty,f^c;R) & \simeq & H^{n-\lambda}(X\setminus f^c,X\setminus f^\infty;R) \end{array}$$

where the horizontal isomorphisms are given by Alexander duality ([39], section 3.3) and the columns are exact. It does follow that i_{c*} is onto if and only if j_c^* is zero. □

Proposition 3.7. *If f is C^2 then $\underline{\gamma}(f,R)$ and $\overline{\gamma}(f,R)$ are critical values of f; they are critical values of f in the sense of Clarke when f is locally Lipschitzien.*

Proof. Take $\underline{\gamma}$ for example: if $c=\underline{\gamma}(f,R)$ is not a critical value then, for small $\epsilon>0$, $f^{c-\epsilon}$ is a deformation retract of $f^{c+\epsilon}$ via the flow of $-\frac{\nabla f}{\|\nabla f\|^2}$, hence $\underline{\gamma}(f,R)\le c-\epsilon$, a contradiction. The same argument applies when f is only locally Lipschitzien, replacing ∇f by a pseudo-gradient. □

Lemma 3.8. *If f is locally Lipschitzien, then*

$$\overline{\gamma}(f,R)=-\underline{\gamma}(-f,R)$$

Proof. Using a (pseudo-)gradient of f as previously, one can see that $X\setminus f^c$ and $(-f)^{-c}$ have the same homotopy type when c is not a critical value of f. Otherwise, choose a sequence of non-critical values $c_n \nearrow c=\overline{\gamma}(f,R)$, then $-c_n\ge\underline{\gamma}(-f,R)$, taking the limit, we have $\overline{\gamma}(f,R)\le-\underline{\gamma}(-f,R)$. Similarly, taking $c'_n\searrow\underline{\gamma}(-f,R)$, then $-c'_n\le\overline{\gamma}(f,R)$, from which the limit gives us the inverse inequality $-\underline{\gamma}(-f,R)\le\overline{\gamma}(f,R)$. □

Now our questions at the beginning are formulated as follows:

(1) Do we have $\underline{\gamma}(f,R)=\overline{\gamma}(f,R)$?

(2) Do $\underline{\gamma}(f,R)$ and $\overline{\gamma}(f,R)$ depend on the coefficient ring R?

Here are two obvious elements for an answer:

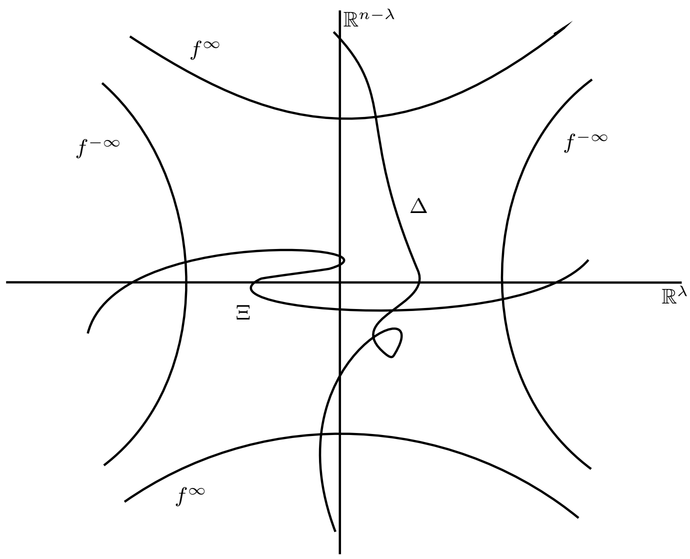
Proposition 3.9. *One has $\underline{\gamma}(f,\mathbb{Z})\ge\underline{\gamma}(f,R)$ and $\overline{\gamma}(f,\mathbb{Z})\le\overline{\gamma}(f,R)$ for every ring R.*

Proof. A simplex σ whose homology class generates $H_\lambda(f^\infty,f^{-\infty};\mathbb{Z})$ induces a simplex whose homology class generates $H_\lambda(f^\infty,f^{-\infty};R)$, hence the first inequality and, mutatis mutandis, the second one. □

Proposition 3.10. *One has $\underline{\gamma}(f,\mathbb{Z})\ge\overline{\gamma}(f,\mathbb{Z})$.*

Proof. As the intersection number of Ξ and Δ is ±1, the support of any descending simplex σ must intersect the support of any ascending simplex τ at some point $\bar{x}$, hence $\max\limits_{x\in|\sigma|}f(x)\ge f(\bar{x})\ge\min\limits_{x\in|\tau|}f(x)$. □

Theorem 3.11. *If $\mathbb{F}$ is a field, then $\underline{\gamma}(f,\mathbb{F})=\overline{\gamma}(f,\mathbb{F})$.*



Proof. By Proposition 3.6, it is enough to prove that

$$\underline{\gamma}(f, \mathbb{F}) = \underline{\alpha}(f, \mathbb{F})$$

Recall that $\underline{\gamma}(f, \mathbb{F})$ (resp. $\underline{\alpha}(f, \mathbb{F})$) is the infimum of the real numbers c such that i_{c*} : $H_\lambda(f^c, f^{-\infty}; \mathbb{F}) \to H_\lambda(f^\infty, f^{-\infty}; \mathbb{F})$ is onto (resp. such that $i_c^* : H^\lambda(f^\infty, f^{-\infty}; \mathbb{F}) \to H^\lambda(f^c, f^{-\infty}; \mathbb{F})$ is nonzero). Now, as $H_\lambda(f^\infty, f^{-\infty}; \mathbb{F})$ is a one-dimensional vector space over $\mathbb{F}$, the linear map i_{c*} is onto if and only if it is nonzero, i.e. if and only if the transposed map i_c^* is nonzero. □

Remark 3.12. This proof is invalid for coefficients in $\mathbb{Z}$ since a $\mathbb{Z}$-linear map to $\mathbb{Z}$, for example $\mathbb{Z} \ni m \to km$, $k \in \mathbb{Z}$, $k > 1$, can be nonzero without being onto; we shall see in Section 3.4 that Theorem 3.11 itself is not true in that case.

Corollary 3.13. *If $\underline{\gamma}(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = \gamma$ then $\underline{\gamma}(f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F}) = \gamma$ for every field $\mathbb{F}$.*

Proof. This follows at once from Theorem 3.11 and Proposition 3.9. □

Corollary 3.14. *Let $\gamma \in \mathbb{R}$ have the following property: there exist both a descending simplex over $\mathbb{Z}$ along which γ is the maximum of f and an ascending simplex over $\mathbb{Z}$ along which γ is the minimum of f. Then, $\underline{\gamma}(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = \underline{\gamma}(f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F}) = \gamma$ for every field $\mathbb{F}$.*

Proof. We have $\underline{\gamma}(f; \mathbb{Z}) \leq \gamma \leq \overline{\gamma}(f; \mathbb{Z})$ by Lemma 3.3 and $\overline{\gamma}(f; \mathbb{Z}) \leq \underline{\gamma}(f; \mathbb{Z})$ by Proposition 3.10, hence our result by Corollary 3.13. □

3.3 Morse complexes and the Barannikov normal form

The previous proof of Theorem 3.11, though simple, is quite algebraic. We now give a more geometric proof, which we find more concrete and illuminating, based on Barannikov's canonical form of Morse complexes. It will provide a good setting for the counterexample in Section 3.4.

First, recall a continuity result for the minmax and maxmin:

Proposition 3.15. *If f and g are two continuous functions quadratic at infinity with the same reference quadratic form, then*

$$\begin{aligned} |\underline{\gamma}(f,R) - \underline{\gamma}(g,R)| &\leq |f-g|_{C^0} \\ |\overline{\gamma}(f,R) - \overline{\gamma}(f,R)| &\leq |f-g|_{C^0}. \end{aligned}$$

Proof. For $f \leq g$, from Lemma 3.3, it is easy to see that $\underline{\gamma}(f) \leq \underline{\gamma}(g)$. In the general case, this implies $\underline{\gamma}(g) \leq \underline{\gamma}(f + |g-f|) \leq \underline{\gamma}(f) + |g-f|_{C^0}$; exchanging f and g, we get $\underline{\gamma}(f) \leq \underline{\gamma}(g) + |f-g|_{C^0}$. ☐

Corollary 3.16. *To prove Theorem 3.11, it suffices to establish it for* excellent Morse *functions $f : X \to \mathbb{R}$, i.e. smooth functions having only non-degenerate critical points, each of which corresponds to a different value of f.*

Proof. By a standard argument, given a non-degenerate quadratic form Q on X, the set of all continuous functions on X equal to Q off a compact subset contains a C^0-dense subset consisting of excellent Morse functions; our result follows by Proposition 3.15. ☐

To prove Theorem 3.11 for excellent Morse functions, we will use Morse homology.

Hypotheses

We consider an excellent Morse function f on X, quadratic at infinity; for each pair of regular values $b < c$ of f, we denote by $f_{b,c}$ the restriction of f to $f^c \cap (-f)^{-b} = \{b \leq f \leq c\}$.

Morse complexes

Let

$$C_k(f_{b,c}) := \{\xi_\ell^k : 1 \leq \ell \leq m_k\}$$

denote the set of critical points of index k of $f_{b,c}$, ordered so that $f(\xi_\ell^k) < f(\xi_m^k)$ for $\ell < m$. Given a generic gradient-like vector field V for f such that (f, V) is Morse-Smale[1], the *Morse complex* of $(f_{b,c}, V)$ over R consists of the free R-modules

$$M_k(f_{b,c}, R) := \{\sum_\ell a_\ell \xi_\ell^k, \quad a_\ell \in R\}$$

together with the boundary operator $\partial : M_k(f_{b,c}, R) \to M_{k-1}(f_{b,c}, R)$ given by

$$\partial \xi_\ell^k := \sum_m \nu_{f,V}(\xi_\ell^k, \xi_m^{k-1}) \xi_m^{k-1}$$

where, with given orientations for the stable manifolds (hence co-orientations for unstable manifolds), $\nu_{f,V}$ is the intersection number of the stable manifold $W^s(\xi_l^k)$ of ξ_l^k and the unstable manifold $W^u(\xi_m^{k-1})$ of ξ_m^{k-1}, i.e. the algebraic number of trajectories of V connecting ξ_ℓ^k and ξ_m^{k-1}; note that

- $\nu_{f,V}(\xi_\ell^k, \xi_m^{k-1})$ is the same for all b, c with $f(\xi_\ell^k), f(\xi_m^{k-1})$ in $[b, c]$;
- $\nu_{f,V}(\xi_\ell^k, \xi_m^{k-1}) \neq 0$ implies $f(\xi_\ell^k) > f(\xi_m^{k-1})$: otherwise, the stable manifold of ξ_m^{k-1} and the unstable manifold of ξ_ℓ^k for V, which cannot be transversal because of their dimensions, would intersect, contradicting the genericity of V.

1. Being Morse-Smale means that the stable and unstable manifolds of all the critical points are transversal.

- $\nu_{f,V}(\xi_l^k, \xi_m^k) = 0$ for two distinct critical points of the same index.

This does define a complex, i.e. $\partial \circ \partial = 0$: see for example [36, 52]. The homology $HM_*(f_{b,c}, R) := H_*(M_*(f_{b,c}, R))$ is called the *Morse homology*[2] of $f_{b,c}$.

Lemma 3.17 (Barannikov,[6]). *If R is a field $\mathbb{F}$, then this boundary operator ∂ has a special kind of Jordan normal form as follows: each $M_k(f_{b,c}, \mathbb{F})$ has a basis*

$$\Xi_\ell^k := \sum_{i \le \ell} \alpha_{\ell,i} \xi_i^k, \quad \alpha_{\ell,\ell} \neq 0 \tag{3.1}$$

such that either $\partial \Xi_\ell^k = 0$ or $\partial \Xi_\ell^k = \Xi_m^{k-1}$ for some m, in which case no $\ell' \neq \ell$ satisfies $\partial \Xi_{\ell'}^k = \Xi_m^{k-1}$. If (Θ_ℓ^k) is another such basis, then $\partial \Xi_\ell^k = \Xi_m^{k-1}$ (resp. 0) is equivalent to $\partial \Theta_\ell^k = \Theta_m^{k-1}$ (resp. 0); in other words, the matrix of ∂ in all such bases is the same.

Proof. We prove existence by induction. Given nonnegative integers k, i with $i < m_k$, suppose that vectors Ξ_q^p of the form (3.1) have been obtained for all (p, q) with either $p < k$, or $p = k$ and $q \le i$, possessing the required property that either $\partial \Xi_q^p = \Xi_{j_p(q)}^{p-1}$ (with $j_p(q) \neq j_p(q')$ for $q \neq q'$) or $\partial \Xi_q^p = 0$. If $\partial \xi_{i+1}^k = 0$ (e.g., when $k = 0$), we take $\xi_{i+1}^k := \Xi_{i+1}^k$ and continue the induction. Otherwise, $\partial \xi_{i+1}^k = \sum \alpha_j \Xi_j^{k-1}$, $\alpha_j \in \mathbb{F}$. Moving all the terms $\Xi_{j_k(q)}^{k-1} = \partial \Xi_q^k, q \le i$ from the right-hand side to the left, we get

$$\partial(\xi_{i+1}^k - \sum_{q \le i} \alpha_{j_k(q)} \Xi_q^k) = \sum_j \beta_j \Xi_j^{k-1}.$$

Let

$$\Xi_{i+1}^k := \xi_{i+1}^k - \sum_{q \le i} \alpha_{j_k(q)} \Xi_q^k.$$

If $\beta_j = 0$ for all j, then $\partial \Xi_{i+1}^k = 0$ and the induction can go on. Otherwise,

$$\partial \Xi_{i+1}^k = \sum_{j \le j_0} \beta_j \Xi_j^{k-1} =: \tilde{\Xi}_{j_0}^{k-1} \text{ with } \beta_{j_0} \neq 0;$$

as $\partial \tilde{\Xi}_{j_0}^{k-1} = \partial\partial \Xi_{i+1}^k = 0$, we can replace $\Xi_{j_0}^{k-1}$ by $\tilde{\Xi}_{j_0}^{k-1}$ and continue the induction[3]. □

Definition 3.18. Under the hypotheses and with the notation of the Barannikov lemma, two critical points ξ_ℓ^k and ξ_m^{k-1} of $f_{b,c}$ are *coupled* if $\partial \Xi_\ell^k = \Xi_m^{k-1}$. A critical point is *free* (over $\mathbb{F}$) when it is not coupled with any other critical point.

In other words, ξ_ℓ^k is free if and only if Ξ_ℓ^k is a cycle of $M_k(f_{b,c}, \mathbb{F})$ but not a boundary, hence the following result:

Corollary 3.19. *For each integer k, the Betti number* $\dim_{\mathbb{F}} HM_k(f_{b,c}, \mathbb{F})$ *is the number of free critical points of index k of $f_{b,c}$ over $\mathbb{F}$.* □

Theorem 3.20. *1. The Barannikov normal form of the Morse complex of $f_{b,c}$ over $\mathbb{F}$ is independent of the gradient-like vector field V.*

2. So is the Morse homology $HM_*(f_{b,c}, R)$; it is isomorphic to $H_*(f^c, f^b; R)$.

2. Morse homology is defined in general for any Morse function without being excellent.
3. Note that if $\mathbb{F}$ was not a field, this would not provide a basis for noninvertible β_{j_0}.

3. *For $b' \leq b < c \leq c'$, the inclusion $i : f^c \hookrightarrow f^{c'}$, restricted to the critical set $C_*(f_{b,c})$, induces a linear map $i_* : M_*(f_{b,c}, R) \to M_*(f_{b',c'}, R)$ such that $\partial \circ i_* = i_* \circ \partial$ and therefore a linear map $i_* : HM_*(f_{b,c}, R) \to HM_*(f_{b',c'}, R)$, which is the usual $i_* : H_*(f^c, f^b; R) \to H_*(f^{c'}, f^{b'}; R)$ modulo the isomorphism (ii).*

Idea of the proof [36]. (1) Connecting two generic gradient-like vector fields V_0, V_1 for f by a generic family, one can prove that each of the Morse complexes defined by V_0 and V_1 is obtained from the other by a change of variables whose matrix is upper-triangular with all diagonal entries equal to 1.

(2) When there is no critical point of f in $\{b \leq f \leq c\}$, both $HM_*(f_{b,c}, R)$ and $H_*(f^c, f^b; R)$ are trivial (the flow of V defines a retraction of f^c onto f^b).

When there is only one critical point ξ of f in $\{b \leq f \leq c\}$, of index λ,

$$HM_k(f_{b,c}, R) \simeq H_k(f^c, f^b; R) \simeq \begin{cases} R, & \text{if } k = \lambda, \\ 0 & \text{otherwise :} \end{cases}$$

the class of ξ obviously generates $HM_\lambda(f_{b,c}, R)$, whereas a generator of $H_\lambda(f^c, f^b; R)$ is the class of a cell of dimension λ, namely the stable manifold of ξ for $V|_{\{b\leq f\leq c\}}$; the isomorphism associates the second class to the first.

In the general case, one can consider a subdivision $b = b_0 < \cdots < b_N = c$ consisting of regular values of f such that each $f_{b_j,b_{j+1}}$ has precisely one critical point. One can show that the boundary operator ∂ of the relative singular homology $\partial : H_{k+1}(f^{b_{i+1}}, f^{b_i}) \to H_k(f^{b_i}, f^{b_{i-1}})$ can be interpreted as the intersection number of the stable manifold of the critical point in $\{b_i \leq f \leq b_{i+1}\}$ and the unstable manifold of that in $\{b_{i-1} \leq f \leq b_i\}$, i.e., their algebraic number of connecting trajectories.

(3) The first claims are easy. The last one follows from what has just been sketched. □

Corollary 3.21. *If f is an excellent Morse function quadratic at infinity, then it has precisely one free critical point ξ over $\mathbb{F}$; its index λ is that of the reference quadratic form Q and*

$$\underline{\gamma}(f, \mathbb{F}) = f(\xi).$$

Proof. Clearly, the dimension of

$$HM_k(f, \mathbb{F}) = HM_k(f_{-\infty,\infty}, \mathbb{F}) \simeq H_k(f^\infty, f^{-\infty}; \mathbb{F}) = H_k(Q^\infty, Q^{-\infty}; \mathbb{F})$$

is 1 if $k = \lambda$ and 0 otherwise. The first two assertions follow by Corollary 3.19. To prove $\underline{\gamma}(f, \mathbb{F}) = f(\xi)$, note that $\underline{\gamma}(f)$ is the infimum of the regular values c of f such that the class of ξ in $HM_\lambda(f_{-\infty,\infty}, \mathbb{F})$ lies in the image of $i_{c*} : HM_\lambda(f_{-\infty,c}, \mathbb{F}) \to HM_\lambda(f_{-\infty,\infty}, \mathbb{F})$; by Theorem 3.20 (iii), which means $c \geq f(\xi)$. □

Proposition 3.22. *The excellent Morse function $-f_{b,c} = (-f)_{-c,-b}$ has the same free critical points over the field $\mathbb{F}$ as $f_{b,c}$.*

Proof. Assuming V fixed, this is essentially easy linear algebra:

- One has $C_k(-f) = C_{n-k}(f)$ and the ordering of the corresponding critical values is reversed. Thus, the lexicographically ordered basis of $M_*(-f)$ corresponding to $(\xi^k_\ell)_{1\leq\ell\leq m_k, 0\leq k\leq n}$ is $(\xi^{n-k}_{m_{n-k}-\ell+1})_{1\leq\ell\leq m_{n-k}, 0\leq k\leq n}$.
- The vector field $-V$ has the same relations with $-f$ as V has with f, hence $\nu_{-f,-V}(\xi^{n-k}_{m_{n-k}-\ell+1}, \xi^{n-(k-1)}_{m_{n-(k-1)}-m+1}) = \nu_{f,V}(\xi^{n-(k-1)}_{m_{n-(k-1)}-m+1}, \xi^{n-k}_{m_{n-k}-\ell+1})$.

That is, the matrix of the boundary operator of $M_*(-f_{b,c})$ in the basis $(\xi^{n-k}_{m_{n-k}-\ell+1})$ is the matrix $\tilde{M}$ obtained from the matrix A of the boundary operator of $M_*(f_{b,c})$ in the basis (ξ^k_ℓ) by symmetry with respect to the second diagonal (i.e. by reversing the order of both the lines and columns of the transpose of A).

Lemma 3.17 can be rephrased as follows: there exists a block-diagonal matrix

$$P = \operatorname{diag}(P_0, \dots, P_n)$$

where each $P_k \in \mathrm{GL}(m_k, \mathbb{F})$ is upper triangular, such that

$$P^{-1}AP = B \tag{3.2}$$

is a Barannikov normal form, meaning the following: the entries of the column of indices ${}^k_\ell$ are 0 except possibly one, equal to 1, which must lie on the line of indices ${}^{k-1}_m$ for some m and be the only nonzero entry on this line. The normal form B is the same for every choice of P and V. Clearly, ξ^k_ℓ is a free critical point of $f_{b,c}$ if and only if both the line and column of indices ${}^k_\ell$ of B are zero.

Equation (3.2) reads

$$\tilde{P}\tilde{A}\tilde{P}^{-1} = \tilde{B}; \tag{3.3}$$

Now, $\tilde{P}^{-1}$ and $\tilde{P} = (\tilde{P}^{-1})^{-1}$ are block diagonal upper triangular matrices whose k^{th} diagonal block lies in $\mathrm{GL}(m_{n-k}, \mathbb{F})$; therefore, by (3.3), as $\tilde{B}$ is a Barannikov normal form for the ordering associated to $-f$, it is *the* Barannikov normal form of the boundary operator of $M_*(-f_{b,c})$, from which our result follows at once. □

Corollary 3.23. *For any excellent Morse function f quadratic at infinity, the sole free critical point of $-f$ over $\mathbb{F}$ is the free critical point ξ of f; hence $\underline{\gamma}(f, \mathbb{F}) = f(\xi) = -(-f)(\xi) = -\underline{\gamma}(-f, \mathbb{F}) = \overline{\gamma}(f, \mathbb{F})$ by Corollary 3.21 and Lemma 3.8, which proves Theorem 3.11.* □

Before we give an example where $\underline{\gamma}(f, \mathbb{Z}) > \overline{\gamma}(f, \mathbb{Z})$, here is a situation where this cannot occur:

Proposition 3.24. *Assume that $M_*(f, \mathbb{Z})$ can be put into Barannikov normal form by a basis change* (3.1) *of the free $\mathbb{Z}$-module $M_*(f, \mathbb{Z})$:*

$$\Xi^k_\ell := \sum_{i \le \ell} \alpha^k_{\ell,i}\xi^k_i, \quad \alpha^k_{\ell,i} \in \mathbb{Z}, \quad \alpha^k_{\ell,\ell} = \pm 1. \tag{3.4}$$

Then, $\underline{\gamma}(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = f(\xi)$, where ξ is the sole free critical point of f over $\mathbb{Z}$.

Proof. We are in the situation of the proof of Proposition 3.22 with $P_k \in \mathrm{GL}(m_k, \mathbb{Z})$, which implies that the Barannikov normal form B of the boundary operator is the same for $\mathbb{Z}$ as for $\mathbb{Q}$; it does follow that there is a unique free critical point ξ of f over $\mathbb{Z}$ (the same as over $\mathbb{Q}$) and that it is the unique free critical point of $-f$ over $\mathbb{Z}$; moreover, the proof of Corollary 3.21 shows that $\underline{\gamma}(f, \mathbb{Z}) = \overline{\gamma}(f, \mathbb{Z}) = f(\xi)$. We conclude as in Corollary 3.23. □

Now that the coefficients are in $\mathbb{Z}$, the classical method of so called *sliding handles* states that, under an additional condition imposed on the index of the change of basis in (3.4), namely $2 \le k \le n-2$, the Barannikov normal form can be realized by a gradient-like vector field for f.

More precisely, let $P : M_*(f) \to M_*(f)$ be a transformation matrix where $P = \operatorname{diag}(P_0, \dots, P_n)$ with each $P_k \in GL(m_k, \mathbb{Z})$ such that $P_k = id$ for $k = 0, 1$ or $n-1, n$, and P_k is upper triangular with ± 1 in the diagonal entries for $2 \leq k \leq n-2$. Then one can construct a gradient-like vector field V' such that, if the matrix of the boundary operator for a given gradient-like vector field V is A, then the matrix for V' is given by $B = P^{-1}AP$.

Roughly speaking, one modifies V, each time for one $i \leq l$, by sliding handle of the stable sphere [4] $S_L(\xi_l^k)$ of ξ_l^k for V such that it sweeps across the unstable sphere $S_R(\xi_i^k)$ of ξ_i^k with indicated intersection number. In other words, $S'_L(\xi_l^k)$ for the resulted V' is the connected sum of $S_L(\xi_l^k)$ and the boundary of a meridian disk of $S_R(\xi_i^k)$ described in section 4.4 of [36]. One may refer to the Basis Theorem (Theorem 7.6) in [52] for a detailed construction of V'.

Remark 3.25 (on the "proof" of Corollary 3.23 in [14])**.** Capitanio uses the following

CRITERION *A critical point ξ of f is free (over $\mathbb{Q}$) if and only if , for any critical point η incident to ξ,there is a critical point ξ', incident to η, such that*

$$|f(\xi') - f(\eta)| < |f(\xi) - f(\eta)|.$$

where fixing a gradient-like vector field V generic for f, two critical points are called incident if their algebraic number of connecting trajectories is nonzero.

Unfortunately, this is not true: one can construct a function $f : \mathbb{R}^{2n} \to \mathbb{R}$, $n \geq 2$, quadratic at infinity with Morse index n, having five critical points, two of index $n-1$ and three of index n, whose gradient vector field V defines the Morse complex

$$\partial\xi_1^n = \xi_2^{n-1}, \quad \partial\xi_2^n = \xi_1^{n-1}, \quad \partial\xi_3^n = 0.$$

This complex can be reformulated into

$$\begin{aligned}
&\partial\xi_1^n = (\xi_2^{n-1} - \xi_1^{n-1}) + \xi_1^{n-1}\\
&\partial(\xi_2^n + \xi_1^n) = (\xi_2^{n-1} - \xi_1^{n-1}) + 2\xi_1^{n-1}\\
&\partial(\xi_3^n + \xi_2^n) = \xi_1^{n-1}
\end{aligned}$$

Hence, for a change of basis

$$\xi_2^{n-1} \mapsto \xi_2^{n-1} - \xi_1^{n-1}, \quad \xi_2^n \mapsto \xi_2^n + \xi_1^n, \quad \xi_n^3 \mapsto \xi_n^3 + \xi_n^2$$

one can construct a gradient-like vector field V' for f by sliding handles, such that

$$\partial\xi_1^n = \xi_2^{n-1} + \xi_1^{n-1}, \quad \partial\xi_2^n = \xi_2^{n-1} + 2\xi_1^{n-1}, \quad \partial\xi_3^n = \xi_1^{n-1}.$$

Obviously, ξ_3^n is the only free critical point, but ξ_2^n satisfies the criterion (with incidences under V'). □

4. The stable and unstable sphere is defined as : $S_L(\xi_l^k) = W^s(\xi_l^k) \cap L$ and $S_R(\xi_i^k) = W^u(\xi_i^k) \cap L$ where $L = f^{-1}(c)$ for some $c \in (f(\xi_i^k), f(\xi_l^k))$.

3.4 An example of Laudenbach

Proposition 3.26. *There exists an excellent Morse function $f : \mathbb{R}^{2n} \to \mathbb{R}$ as follows:*

1. *it is quadratic at infinity and the reference quadratic form has index and coindex $n > 1$;*
2. *it has exactly five critical points: three of index n, one of index $n-1$ and one of index $n+1$;*
3. *its Morse complex over $\mathbb{Z}$ is given by*

$$\begin{aligned}
&\partial\xi_1^{n-1} = 0\\
&\partial\xi_1^{n} = \xi_1^{n-1}, \quad \partial\xi_2^{n} = -2\xi_1^{n-1}, \quad \partial\xi_3^{n} = -\xi_1^{n-1}\\
&\partial\xi_1^{n+1} = \xi_2^{n} - 2\xi_3^{n},
\end{aligned} \tag{3.5}$$

hence, for any field $\mathbb{F}_2$ of characteristic 2 and any field $\mathbb{F}$ of characteristic $\neq 2$,

$$\underline{\gamma}(f,\mathbb{Z}) = \underline{\gamma}(f,\mathbb{F}_2) = \overline{\gamma}(f,\mathbb{F}_2) = f(\xi_3^n) \ > \ f(\xi_2^n) = \underline{\gamma}(f,\mathbb{F}) = \overline{\gamma}(f,\mathbb{F}) = \overline{\gamma}(f,\mathbb{Z}). \tag{3.6}$$

Proof that (3.5) *implies* (3.6). The Morse complex of f over $\mathbb{F}_2$ writes

$$\begin{aligned}
&\partial\xi_1^{n-1} = 0\\
&\partial\xi_1^{n} = \xi_1^{n-1}, \quad \partial\xi_2^{n} = 0, \quad \partial(\xi_3^{n} + \xi_1^{n}) = 0\\
&\partial\xi_1^{n+1} = \xi_2^{n},
\end{aligned}$$

implying that ξ_3^n is the only free critical point, hence, by Corollary 3.21,

$$\underline{\gamma}(f,\mathbb{F}_2) = \overline{\gamma}(f,\mathbb{F}_2) = f(\xi_3^n);$$

as $\underline{\gamma}(f,\mathbb{Z}) \geq \underline{\gamma}(f,\mathbb{F}_2)$ by Proposition 3.9 and $\underline{\gamma}(f,\mathbb{Z}) \leq f(\xi_3^n)$, we do have

$$\underline{\gamma}(f,\mathbb{Z}) = f(\xi_3^n).$$

Similarly (keeping the numbering of the critical points defined by f) the Morse complex of $-f$ over $\mathbb{F}$ has the Barannikov normal form

$$\begin{aligned}
&\partial(-2\xi_1^{n+1}) = 0\\
&\partial\xi_3^{n} = -2\xi_1^{n+1}, \quad \partial(\xi_2^{n} + \frac{1}{2}\xi_3^{n}) = 0, \quad \partial(-\xi_3^{n} - 2\xi_2^{n} + \xi_1^{n}) = 0\\
&\partial\xi_1^{n-1} = -\xi_3^{n} - 2\xi_2^{n} + \xi_1^{n},
\end{aligned}$$

showing that the free critical point is ξ_2^n; hence, by Corollary 3.21 and Proposition 3.22,

$$\overline{\gamma}(f,\mathbb{F}) = \underline{\gamma}(f,\mathbb{F}) = f(\xi_2^n);$$

finally, as we have $\overline{\gamma}(f,\mathbb{Z}) \leq \overline{\gamma}(f,\mathbb{F})$ by Proposition 3.9, and $\overline{\gamma}(f,\mathbb{Z}) \geq f(\xi_1^n)$, we should prove $\overline{\gamma}(f,\mathbb{Z}) > f(\xi_1^n)$, which is obvious since ξ_1^n and ξ_1^{n+1} are boundaries in $M_*(-f,\mathbb{Z})$.

How to construct such a function f. It is easy to construct a function $f_0 : \mathbb{R}^{2n} \to \mathbb{R}$ with properties (1) and (2) required in the proposition and whose gradient vector field V_0 provides a Morse complex given by

$$\begin{aligned}
&\partial\xi_1^{n-1} = 0,\\
&\partial\xi_1^{n} = \xi_1^{n-1}, \quad \partial\xi_2^{n} = 0, \quad \partial\xi_3^{n} = 0\\
&\partial\xi_1^{n+1} = \xi_3^{n}.
\end{aligned}$$

For a change of basis

$$\xi_2^n \mapsto \xi_2^n - \xi_1^n, \quad \xi_3^n \mapsto \xi_3^n - 2(\xi_2^n - \xi_1^n)$$

one can construct a gradient-like vector field V' for f_0 by sliding handles, such that

$$\begin{aligned} &\partial\xi_1^{n-1} = 0 \\ &\partial\xi_1^n = \xi_1^{n-1}, \quad \partial\xi_2^n = -\xi_1^{n-1}, \quad \xi_3^n = -2\xi_1^{n-1} \\ &\partial\xi_1^{n+1} = -2\xi_2^n + \xi_3^n \end{aligned}$$

Since (f_0, V') is Morse-Smale, the invariant manifolds of those critical points of the same index are disjoint, hence one can modify f_0 to f such that

- f has the same critical points of f_0;
- the ordering of critical points for f is $f(\xi_2^n) > f(\xi_3^n) > f(\xi_1^n)$,
- V' is a gradient-like vector field for f.

This can be realized by the preliminary rearrangement theorem (Theorem 4.1) in [52].

In other words, we have made a change of critical points $\xi_2^n \leftrightarrow \xi_3^n$, hence obtain the required Morse complex in the proposition.

Question 3.27. *For a generating family $S(x, \eta)$ of a Lagrangian submanifold $L \subset T^*M$ Hamiltonian isotopy to the zero section, where each $S_x : \mathbb{R}^k \to \mathbb{R}$ is quadratic at infinity, do we have $\underline{\gamma}(S_x, \mathbb{Z}) = \overline{\gamma}(S_x, \mathbb{Z})$ for every $x \in M$.*

3.5 On the product formula for minmax

In Chapter 2, we have proved in Lemma 2.40 a product formula for the minmax using the equivalence of minmax and maxmin for coefficients in $\mathbb{Z}_2$. In this section, we give anther proof based on the Barannikov normal form for coefficients in any field, which give us an acess to a counterexample for coefficients in $\mathbb{Z}$.

We will first describe the *Künneth formula* in Morse language. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be two Morse functions, and let V_f and V_g their corresponding pseudo-gradients satisfying the Morse-Smale condition. Let

$$f \oplus g : X \times Y \to \mathbb{R}, \quad (f \oplus g)(x, y) = f(x) + g(y).$$

It is a Morse function on $X \times Y$ with pseudo-gradient $V_{f\oplus g} = V_f \times V_g$, and the set of critical points of $f \oplus g$ is

$$C_k(f \oplus g) = \bigcup_{i+j=k} C_i(f) \times C_j(g).$$

Note that since $V_{f\oplus g}$ is split, so is its flow, hence the number of connecting trajectories ν of is given as follows: abbreviating $\nu_{f,V}$ by ν_f, if $(\xi, \eta) \in C_k(f \oplus g)$ and $(\xi', \eta') \in C_{k-1}(f \oplus g)$,

$$\nu_{f\oplus g}((\xi, \eta), (\xi', \eta')) = \pm\nu_f(\xi, \xi') \times \nu_g(\eta, \eta') = \begin{cases} \pm\nu_f(\xi, \xi') & \text{if } \eta = \eta', \\ \pm\nu_g(\eta, \eta') & \text{if } \xi = \xi', \\ 0 & \text{otherwise} \end{cases}$$

because V_f and V_g are Morse-Smale.

Let R be a commutative ring with unit, and let $M_k(f\oplus g, R)$ denote the Morse complex. We use the tensor product for chain complexes

$$(M_*(f,R)\otimes M_*(g,R))_k = \bigoplus_{i+j=k} M_i(f,R)\otimes M_j(g,R)$$

together with the boundary operator

$$\partial(\sigma_i\otimes\tau_j) = \partial_{V_f}\sigma_i\otimes\tau_j + (-1)^i\sigma_i\otimes\partial_{V_g}\tau_j, \quad \text{for}\, \sigma_i\otimes\tau_j\in M_i(f,R)\times M_j(g,R).$$

With a proper cohererent orientation, there is an isomorphism of Morse complexes

$$(M_*(f,R)\otimes M_*(g,R),\partial) \xrightarrow{\simeq} (M_*(f\oplus g,R),\partial_{V_{f\oplus g}})$$

See section 5.3 in [56] for reference.

Now suppose that f,g are Morse functions as before, and in addition, quadratic at infinity. If λ_f and λ_g denote the indices of the corresponding quadratic forms, then $\lambda_{f\oplus g} = \lambda_f + \lambda_g$.

Lemma 3.28. *We have an isomorphism of homology groups*

$$H_{\lambda_f}(f;R)\otimes H_{\lambda_g}(g;R) \simeq H_{\lambda_{f\oplus g}}(f\oplus g;R).$$

Proof. By the Künneth formula, we have a short exact sequence

$$0\to \bigoplus_{i+j=k} H_i(f,R)\otimes H_j(g,R) \to H_k(f\oplus g,R) \to \bigoplus_{i+j=k-1} \mathrm{Tor}_1(H_i(f,R),H_j(g,R)) \to 0.$$

For $k=\lambda_f+\lambda_g$, we conclude since $H_i(f,R)$ and $H_j(g,R)$ are zero if $i\neq\lambda_f$ and $j\neq\lambda_g$ respectively. □

Proposition 3.29. *For a field $\mathbb{F}$, we have the product formula for minmax*

$$\underline{\gamma}(f\oplus g,\mathbb{F}) = \underline{\gamma}(f,\mathbb{F}) + \underline{\gamma}(g,\mathbb{F})$$

Proof. For simplicity, we write $\lambda_f = n$ and $\lambda_g = m$. Equip $M_i(f)$ and $M_j(g)$ with Barannikov normal bases $\{\Xi^i\}$ and $\{\Theta^j\}$ respectively (where we have omitted the subscript enumerating the members of each basis). Let ξ and θ be are the unique free points of f and g respectively, and let $\Xi := \Xi^n_{\ell_1}$, $\Theta := \Theta^m_{\ell_2}$ be the corresponding elements of the normal bases. Then, by Künneth's formula, $\Xi\otimes\Theta$ is a generator for $H_{n+m}(f\oplus g,\mathbb{F})$ and all generators have the form $\sigma = \Xi\otimes\Theta + \partial\sigma^{n+m+1}$, with $\sigma^{n+m+1}\in\oplus_{i+j=n+m+1}M_i(f)\otimes M_j(g)$, hence the inequality

$$\underline{\gamma}(f\oplus g) \le f(\xi)+g(\theta) = \underline{\gamma}(f)+\underline{\gamma}(g).$$

To obtain the reverse inequality, we write the generators as

$$\begin{aligned}\sigma &= \Xi\otimes\Theta + \alpha\partial\Xi^{n+1}\otimes\Theta^m + \beta\Xi^n\otimes\partial\Theta^{m+1} + \text{other terms}\\ &= \Xi\otimes\Theta + \sum_{\ell\neq\ell_1}\alpha_{\ell,h}\Xi^n_\ell\otimes\Theta^m_h + \sum_{\ell\neq\ell_2}\beta_{h,\ell}\Xi^n_h\otimes\Theta^m_\ell + \text{other terms.}\end{aligned}$$

By the construction of the Barannikov normal form, since $\partial\Xi^n_{\ell_1} = \partial\Theta^m_{\ell_2} = 0$, the terms Ξ^n_ℓ with $\ell\neq\ell_1$ do not contain $\xi=\xi^n_{\ell_1}$ and the terms Θ^m_ℓ with $\ell\neq\ell_2$ do not contain $\theta=\theta^m_{\ell_2}$. Hence σ always contains the term $\xi\otimes\theta$ which can not be annulated, yielding $\underline{\gamma}(f\oplus g) = \min\max_\sigma f\oplus g \ge f(\xi)+g(\theta) = \underline{\gamma}(f)+\underline{\gamma}(g)$ as required. □

Now we come to the question of the product formula for minmax with coefficients in $\mathbb{Z}$. As the first part of the previous proof does not use the fact that $\mathbb{F}$ is a field, we have

$$\underline{\gamma}(f \oplus g, \mathbb{Z}) \leq \underline{\gamma}(f, \mathbb{Z}) + \underline{\gamma}(g, \mathbb{Z}).$$

The reverse inequality is not true in general. A counterexample is given as follows :

Proposition 3.30. *If f is the Morse function of Proposition 3.26 and $g = -f$, then*

$$\underline{\gamma}(f, \mathbb{Z}) + \underline{\gamma}(g, \mathbb{Z}) = f(\xi_3^n) - f(\xi_2^n) > 0 \quad \textit{and} \quad \underline{\gamma}(f \oplus g, \mathbb{Z}) \leq 0.$$

In particular, $\underline{\gamma}(f \oplus g, \mathbb{Z}) < \underline{\gamma}(f, \mathbb{Z}) + \underline{\gamma}(g, \mathbb{Z})$.

Proof. We have proved in Proposition 3.26 that

$$\underline{\gamma}(f, \mathbb{Z}) = f(\xi_3^n), \quad \underline{\gamma}(g, \mathbb{Z}) = -\overline{\gamma}(f, \mathbb{Z}) = -f(\xi_2^n)$$

Moreover, $[\xi_3^n + \xi_1^n]$ is a generator of $HM_n(f, \mathbb{Z})$ and $[2\xi_2^n + \xi_3^n]$ is a generator of $HM_n(f, \mathbb{Z})$.

Now by the Künneth formula, $[(\xi_3^n + \xi_1^n) \otimes (2\xi_2^n + \xi_3^n)]$ is a generator of $HM_{2n}(f \oplus g, \mathbb{Z})$. Let us consider the cycle σ:

$$\begin{aligned}
\sigma &= (\xi_3^n + \xi_1^n) \otimes (2\xi_2^n + \xi_3^n) + \partial(\xi_1^{n+1} \otimes \xi_2^n) \\
&= (\xi_3^n + \xi_1^n) \otimes (2\xi_2^n + \xi_3^n) + \partial\xi_1^{n+1} \otimes \xi_2^n + (-1)^n \xi_1^{n+1} \otimes \partial\xi_2^n \\
&= \xi_3^n \otimes \xi_3^n + 2\xi_1^n \otimes \xi_2^n + \xi_1^n \otimes \xi_3^n + \xi_2^n \otimes \xi_2^n + (-1)^n \xi_1^{n+1} \otimes \xi_1^{n+1}
\end{aligned}$$

Since $f(\xi_3^n) > f(\xi_2^n) > f(\xi_1^n)$, we have

$$\max_{\sigma} f \oplus g = \max\{f(\xi_3^n) - f(\xi_3^n), f(\xi_1^n) - f(\xi_2^n), f(\xi_1^n) - f(\xi_3^n), \cdots\} = 0$$

hence $\underline{\gamma}(f \oplus g, \mathbb{Z}) \leq 0 < f(\xi_3^n) - f(\xi_2^n) = \underline{\gamma}(f, \mathbb{Z}) - \overline{\gamma}(f, \mathbb{Z})$. □

Remark 3.31. For any f quadratic at infinity, we have

$$\underline{\gamma}(f \oplus (-f), \mathbb{Z}) \geq \underline{\gamma}(f \oplus (-f), \mathbb{F}) = \underline{\gamma}(f, \mathbb{F}) + \underline{\gamma}(-f, \mathbb{F}) = \underline{\gamma}(f, \mathbb{F}) - \overline{\gamma}(f, \mathbb{F}) = 0$$

Question 3.32. For any function f quadratic at infinity, have we $\underline{\gamma}(f \oplus (-f), \mathbb{Z}) = 0$? Hence, if $\underline{\gamma}(f, \mathbb{Z}) \neq \overline{\gamma}(f, \mathbb{Z})$, i.e., $\underline{\gamma}(f, \mathbb{Z}) > \overline{\gamma}(f, \mathbb{Z}) = -\underline{\gamma}(-f, \mathbb{Z})$ by Proposition 3.10, then

$$\underline{\gamma}(f \oplus (-f), \mathbb{Z}) < \underline{\gamma}(f, \mathbb{Z}) + \underline{\gamma}(-f, \mathbb{Z}).$$

Appendix A

Lipschitz critical point theory

We consider a real locally Lipschitz function f on[1] $X := \mathbb{R}^k$.

Definition A.1. The *Clarke generalized derivative* $\partial f(a)$ of f at $a \in X$ is the convex subset $\partial f(a)$ of $X^* = T_x^* X$ defined as follows: by Rademacher's theorem, the set $\mathrm{dom}(df)$ of differentiability points of f is dense in X; if $df := \{(x, df(x)) : x \in \mathrm{dom}(df)\}$, we let

$$\partial f(a) := \mathrm{co}\{y \in X^* : (a, y) \in \overline{df}\},$$

where co stands for the convex hull; in other words, $\partial f(a)$ is the convex hull of the set of limits of convergent sequences $df(x_n)$ with $\lim x_n = a$. As $|df(x)|$ is bounded by the local Lipschitz constant of f for x close to a, every sequence $df(x_n)$ with $\lim x_n = a$ is bounded and therefore has a convergent subsequence, implying

$$\forall a \in X \quad \partial f(a) \neq \emptyset;$$

moreover, $\partial f(a)$ *is compact*, being the convex hull of a compact subset[2]. The subset

$$\partial f := \{(x, y) | y \in \partial f(x), x \in X\}$$

is a generalized version of the *enlarged pseudograph* defined for semi-concave functions in [2], where the *pseudograph* is $df := \{(x, df(x)) | x \in \mathrm{dom}(df)\}$. In simple one-dimensional cases, it is obtained by adding a vertical segment to df where f is not differentiable:

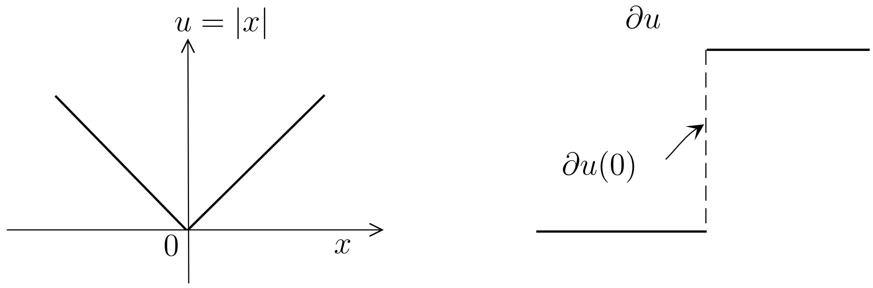


Figure A.1

1. The theory extends to reflexive Banach spaces [16].
2. Caratheodory proved that it is the set of convex combinations of $k+1$ points of $\{y \in X^* : (a, y) \in \overline{df}\}$.

Remark A.2. The set $\partial f(x)$ consists of a single point if and only if f is "C^1 at x with respect to the set where it is differentiable".

Example A.3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 \sin \frac{1}{x}$, then f is differentiable everywhere, C^1 except at $x = 0$, and $df(0) = 0$ whereas $\partial f(0) = [-1, 1]$.

Unfortunately, even though a Lipschitz function is differentiable almost everywhere, it may not be C^1 almost everywhere:

Example A.4. If $\chi_A : \mathbb{R} \to \{0, 1\}$ is the characteristic function of a Cantor set $A \subset \mathbb{R}$ of positive measure, then

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \int_0^x \chi_A(t)dt$$

is Lipschitzian but not C^1 almost everywhere: indeed $f'(x) = \chi_A(x)$ almost everywhere and, as A has no interior point, χ_A is not continuous at points $x \in A$.

For the relation of partial derivative, if $f : X \times Y \to \mathbb{R}$ is Lispschitz, in general, we do not have the relation $\partial f(x, y) = \partial_x f(x, y) \times \partial_y f(x, y)$.

Example A.5. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $f(x, y) = |x - y|$. Then $\partial_1 f(0, 0) \times \partial_2 f(0, 0) = [-1, 1] \times [-1, 1]$. But

$$\begin{aligned}
\partial f(0,0) &= \operatorname{co}\{\lim df(x_n, y_n),\ (x_n, y_n) \to 0\} \\
&= \operatorname{co}\{\lim d_1 f(x_n, y_n) \times d_2 f(x_n, y_n),\ (x_n, y_n) \to 0\} \\
&= \operatorname{co}\{(a, -a),\ a = \pm 1\} = \{(a, -a) \in \mathbb{R}^2, a \in [0, 1]\}
\end{aligned}$$

Example A.6. For the generating family $S(x, \eta)$ defined in (1.4) or (1.5) with initial function v Lipschitz, we have $\partial_x S(x, \eta) = \partial_x S(x, \eta) \times \partial_\eta S(x, \eta)$. Indeed, if we write $S(x, \eta) = v(x_0) + f(x, \eta)$, where v is considered as a function of (x, η) and f is C^2. It is easy to see that

$$\partial S(x, \eta) = \partial v(x_0) + \partial f(x, \eta) = \partial_x f(x, \eta) \times \partial_\eta S(x, \eta) = \partial_x S(x, \eta) \times \partial_\eta S(x, \eta).$$

since $\partial v(x_0) = \{0\} \times \partial_\eta v(x_0)$.

Proposition A.7. *The set-valued function $x \mapsto \partial f(x)$ is* upper semi-continuous: *for every convergent sequence $(x_n, y_n) \to (x, y)$ with $y_n \in \partial f(x_n)$, one has $y \in \partial f(x)$. In other words, ∂f is closed.*

Proof. Each y_n writes $y_n = t_{n,1}v_{n,1} + \cdots + t_{n,k+1}v_{n,k+1}$ with $t_{n,i} \in [0, 1]$, $t_{n,1} + \cdots + t_{n,k+1} = 1$ and $(x_n, v_{n,i}) \in \overline{df}$. As f is locally Lipschitzian, there is a compact subset $K \subset X^*$ containing every $v_{n,i}$ for n large enough; hence, extracting subsequences, we may assume that the sequences $(v_{n,i})_n$, and $(t_{n,i})_n$ converge respectively to $v_i \in K$ and $t_i \in [0, 1]$. As there are points of df arbitrarily close to each $(x_n, v_{n,i})$, we may also assume $(x_n, v_{n,i}) \in df$, hence $v_i \in \partial f(x)$ and therefore $y = t_1 v_1 + \cdots + t_{k+1} v_{k+1} \in \partial f(x)$. □

Proposition A.8. *For any $x, y \in X$, one has*

$$f(y) - f(x) \in co\, \partial f([x, y])(y - x)$$

where the right-hand side denotes the convex hull of all points of the form $z(y - x)$ with $z \in \partial f(u)$ for some u in the line segment $[x, y]$.

Proof. It suffices to prove the inclusion for points y having the property that the segment $[x, y]$ meet dom(df) in a set of full one dimensional measure. Almost all y has this property, and the general case will follow by a limiting argument bases on the continuity of f and upper semicontinuity of ∂f. For such y, we may write

$$f(y) - f(x) = \int_0^1 df(x + t(y - x))(y - x)dt$$

which directly expresses $f(y) - f(x)$ as a convex combination of points from $\partial f([x, y])(y - x)$. □

Corollary A.9. *Let $\phi : [0, 1] \to X$ be C^1, then the function $h = f \circ \phi$ is differentiable almost everywhere and*

$$h'(t) \leq \max\{z\phi'(t) | z \in \partial f(\phi(t))\}$$

Proof. The function h is locally Lipschitz so is differentiable almost everywhere. Suppose that it is differentiable at $t = t_0$, then

$$\begin{aligned}
h'(t_0) &= \lim_{\lambda \to 0} [f(\phi(t_0 + \lambda)) - f(\phi(t_0))]/\lambda \\
&= \lim_{\lambda \to 0} [f(\phi(t_0) + \phi'(t_0)\lambda + o(\lambda)) - f(\phi(t_0))]/\lambda \\
&= \lim_{\lambda \to 0} [f(\phi(t_0) + \phi'(t_0)\lambda) - f(\phi(t_0))]/\lambda \quad \text{by the Lipschitz condition} \\
&\leq \lim_{\lambda \to 0} \max\{z\phi'(t_0) | z \in \partial f([\phi(t_0), \phi(t_0) + \phi'(t_0)\lambda])\} \\
&= \max\{z\phi'(t_0) | z \in \partial f(\phi(t_0))\}
\end{aligned}$$

□

Definition A.10. A point $x \in X$ is called a *critical point* of f if $0 \in \partial f(x)$; the number $f(x)$ is then called a *critical value* of f. By Proposition A.7, the *critical set* Crit(f) of f, consisting of its critical points, is closed in X.

Setting

$$\lambda(x) := \min_{w \in \partial f(x)} |w|_{X^*},$$

we say that f satisfies the *Palais-Smale condition* (P.S.) if every sequence (x_n) along which $f(x_n)$ is bounded, and such that $\lambda(x_n)$ goes to 0, possesses a convergent subsequence—whose limit is a critical point of f by Proposition A.7, as there is a sequence $y_n \in \partial f(x_n)$ converging to 0.

Example A.11. The P.S. condition is satisfied when Lip$(f - Q) < \infty$ for some *nondegenerate* quadratic form Q on X; moreover, in that case, Crit(f) is *compact.*

Proof. Indeed, if $\psi := f - Q$, each subset $\partial f(x) = \partial\psi(x) + dQ(x)$ consists of vectors whose norm is at least $|dQ(x)| - \text{Lip}(\psi)$, hence $\lambda(x) \geq |dQ(x)| - \text{Lip}(\psi)$, which tends to $+\infty$ when $|x| \to \infty$; therefore, there exists $R > 0$ such that every sequence (x_n) with $\lim \lambda(x_n) = 0$ satisfies $|x_n| \leq R$ for all large enough n, implying both the P.S. condition and the compactness of Crit(f). □

Proposition A.12. *When f satisfies the P.S. condition,* Crit$(f) \cap f^{-1}(K)$ *is compact for every compact $K \subset \mathbb{R}$.*

Proof. For each sequence x_n in Crit$(f) \cap f^{-1}(K)$, the sequence $f(x_n) \in K$ is bounded and the sequence $\lambda(x_n) = 0$ converges to 0 hence, by the P.S. condition, x_n has a convergent subsequence, whose limit lies in the closed subset Crit$(f) \cap f^{-1}(K)$. □

Lemma A.13 (existence of pseudo-gradients). *Let B be an open subset of X; if one has $\lambda(x) \geq b > 0$ for all $x \in B$, then there exists a C^∞ vector field v on B such that*

$$|v(x)| \leq 1 \quad \textit{and} \quad x^* v(x) > b/2 \quad \textit{for all} \quad x^* \in \partial f(x),\ x \in B.$$

Proof. For each $a \in B$, there exists $v_a \in X$ such that $|v_a| = 1$ and $x^* v_a \geq \lambda(a)$ for all $x^* \in \partial f(a)$: just take $v_a = w_a/|w_a|$, where w_a is the orthogonal projection of the origin onto the image of $\partial f(a)$ by the isomorphism $X^* \to X$ defined by the scalar product.

Hence, by proposition B5, there is an open subset $V_a \ni a$ of B such that $df(x)v_a > 2\lambda(a)/3 \geq 2b/3$ for all $x \in V_a \cap \mathrm{dom}(df)$ and therefore $x^* v_a \geq 2b/3 > b/2$ for all $x^* \in \partial f(x)$, $x \in V_a$.

Denoting by (θ_i) a smooth partition of unity subordinate to a locally finite cover (V_{a_i}) of B extracted from the cover $(V_a)_{a \in B}$, the vector field $v : B \to X$ can be defined by

$$v(x) := \sum \theta_i(x) v_{a_i}$$

where, as usual, the (finite) sum is over those i's for which $x \in V_{a_i}$. □

Theorem A.14 (Deformation Lemma, straightforward part). *Suppose f satisfies the P.S. condition and let $f^c := \{x | f(x) \leq c\}$ for each $c \in \mathbb{R}$. If c is not a critical value of f, then there exist $\epsilon > 0$ and a bounded smooth vector field V on X equal to 0 off $f^{c+2\epsilon} \smallsetminus f^{c-2\epsilon}$, and whose flow φ_V^t satisfies $\varphi_V^1(f^{c+\epsilon}) \subset f^{c-\epsilon}$.*

Proof. By the P.S. condition, there exist $\epsilon > 0$ and $b > 0$ such that $\lambda(x) \geq b$ in the open subset $B := \{x | c + 3\epsilon > f(x) > c - 3\epsilon\}$; if v is a vector field as in Lemma A.13 and χ a nonnegative smooth function equal to 0 off $\{x | c + 2\epsilon > f(x) > c - 2\epsilon\}$ and reaching its maximum $4\epsilon/b$ for $c + \epsilon \geq f(x) \geq c - \epsilon$, we claim that our requirements are fulfilled by

$$V(x) := \begin{cases} -\chi(x)v(x) & \text{for } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

by Corollary A.9, the Lipschitz function $t \mapsto f \circ \varphi_V^t(x)$ satisfies almost everywhere

$$\frac{d}{dt} f \circ \varphi_V^t(x) \leq \begin{cases} \max\{zV(\varphi_V^t(x)) | z \in \partial f(\varphi_V^t(x))\} \leq -\frac{b}{2}\chi(\varphi_V^t(x)) & \text{for } \varphi_V^t(x) \in B, \\ 0 & \text{otherwise,} \end{cases}$$

it is nonincreasing and, for $x \in f^{c+\epsilon} \smallsetminus f^{c-\epsilon}$ and $t \geq 0$, one has

$$f(\varphi_V^t(x)) - f(x) \leq -\tfrac{b}{2}\tfrac{4\epsilon}{b} t = -2\epsilon t$$

as long as $\varphi_V^t(x)$ remains in $f^{c+\epsilon} \smallsetminus f^{c-\epsilon}$, hence our result. □

Theorem A.15 (Deformation Lemma, subtle part). *Suppose f satisfies the P.S. condition. If $c \in \mathbb{R}$ is a critical value of f and N* any *neighbourhood of $K_c := \mathrm{Crit}(f) \cap f^{-1}(c)$, then there exist $\epsilon > 0$ and a bounded smooth vector field V on X equal to 0 off $f^{c+2\epsilon} \smallsetminus f^{c-2\epsilon}$, whose flow φ_V^t satisfies $\varphi_V^1(f^{c+\epsilon} \smallsetminus N) \subset f^{c-\epsilon}$.*

Proof. Choose $\delta > 0$ so that N contains the closed 4δ-neighbourhood $B_{4\delta}(K_c)$ of K_c.

Lemma. *There exist $\epsilon > 0$, $b > 0$ such that $\lambda(x) \geq b$ for all $x \in (f^{c+3\epsilon} \smallsetminus f^{c-3\epsilon}) \smallsetminus B_\delta(K_c)$.*

Indeed, otherwise, there would exist a sequence (x_n) in $X \smallsetminus B_\delta(K_c)$ with $|f(x_n) - c| \leq \frac{3}{n}$ and $\lambda(x_n) < \frac{1}{n}$; by the P.S. condition, it would have a convergent subsequence, whose limit would be a critical point x with $f(x) = c$ and $d(x, K_c) \geq \delta$, a contradiction. ⊟

We can of course take

$$\epsilon \leq \frac{b}{4}\delta.$$

The open subset $B := \{x \in X | c - 3\epsilon < f(x) < c + 3\epsilon \text{ and } d(x, K_c) > \delta\}$ satisfies the hypotheses of Lemma A.13; if v denotes the ensuing pseudo-gradient vector field, we claim that V can be defined by

$$V(x) := \begin{cases} -\chi(x)\psi(x)v(x) & \text{for } x \in B, \\ 0 & \text{otherwise,} \end{cases}$$

where the smoth real functions $\chi : X \to [0, \frac{4\epsilon}{b}]$ and $\psi : X \to [0, 1]$ satisfy

$$\begin{aligned} \chi(x) &= \begin{cases} \frac{4\epsilon}{b} & \text{in } f^{c+\epsilon} \smallsetminus f^{c-\epsilon}, \\ 0 & \text{off } f^{c+2\epsilon} \smallsetminus f^{c-2\epsilon} \end{cases} \\ \psi(x) &= \begin{cases} 0 & \text{in } B_{2\delta}(K_c), \\ 1 & \text{off } B_{3\delta}(K_c). \end{cases} \end{aligned}$$

Indeed, for $x \in f^{c+\epsilon} \smallsetminus B_{4\delta}(K_c)$, two cases can occur:

- for $x \in f^{c-\epsilon}$, the nonincreasing function[3] $t \mapsto f \circ \varphi_V^t(x)$ takes its values in $f^{c-\epsilon}$ for $t \geq 0$;
- otherwise, for $t \in [0, 1]$, the inequality $|V(x)| \leq \frac{4\epsilon}{b} \leq \delta$ yields

  $$|\varphi_V^t(x) - x| \leq \delta t, \text{ hence } \varphi_V^t(x) \in f^{c+\epsilon} \smallsetminus B_{3\delta}(K_c);$$

  it follows that, as long as $f \circ \varphi_V^t(x) \geq c - \epsilon$, one has $V\big(\varphi_V^s(x)\big) = \frac{4\epsilon}{b} v\big(\varphi_V^s(x)\big)$ for $0 \leq s \leq t$ and therefore, as in the proof of Theorem A.14,

  $$f(x) - f\big(\varphi_V^t(x)\big) = f(x) - f\Big(\varphi_v^{-\frac{4\epsilon}{b}t}(x)\Big) \geq \tfrac{b}{2}\big(\tfrac{4\epsilon}{b}t\big) = 2\epsilon t.$$

Since $t \mapsto f\big(\varphi_V^t(x)\big)$ is nonincreasing, this implies $\varphi_V^1(x) \in f^{c-\epsilon}$. □

Lemma A.16. *If $f : X \to \mathbb{R}$ is a Lipschitz function, $F : X \to X$ a C^1 diffeomorphism, then*

$$\partial(f \circ F)(x) = \partial f(F(x)) \circ dF(x) := \{dF(x)(\xi), \xi \in \partial f(F(x))\}$$

Proof. Let $h(x) = f \circ F(x)$, then $h : X \to \mathbb{R}$ is Lipschitz hence differentiable almost everywhere. Firstly, we claim that, if h is differentiable at x, then, $dh(x) = dF(x)df(F(x))$. Indeed, for any $v \in X^*$, we have

$$\begin{aligned} dh(x)(v) &= \lim_{t\to 0}[f(F(x + tv)) - f(F(x))]/t \\ &= \lim_{t\to 0}[f(F(x) + tdF(x)(v)) - f(F(x))]/t \\ &= \lim_{t\to 0}[f(F(x) + tu) - f(F(x))]/t \end{aligned}$$

Since $dF(x)$ is a bijective linear map, we get that f is differentiable at $F(x)$ by definition.

$$\begin{aligned} \partial(f \circ F)(x) &= \mathrm{co}\{\lim d(f \circ F)(x_n), x_n \to x\} = \mathrm{co}\{\lim dF(x_n)(df(F(x_n))), x_n \to x\} \\ &= \mathrm{co}\{\lim dF(x)(df(F(x_n))), x_n \to x\} \\ &\subset \mathrm{co}\{\lim dF(x)(df(y_n)), y_n \to x\} \\ &= \partial f(F(x)) \circ dF(x) \end{aligned}$$

The inclusion becomes equality since F is onto. □

3. See the proof of Theorem A.14.

Remark A.17. In fact, to prove the inclusion "$\subset$", one dot not need F to be diffeomorphism, see Theorem 2.3.10 in [27] for a general statement.

Lemma A.18. *If $f, g : X \to \mathbb{R}$ are Lipschitz functions, then*

$$\partial(fg)(x) \subset f(x)\partial g(x) + g(x)\partial f(x)$$

Proof. By definition,

$$\begin{aligned}
\partial(fg)(x) &= \mathrm{co}\{\lim d(fg)(x_n),\, x_n \to x\} \\
&= \mathrm{co}\{\lim (f(x_n)dg(x_n) + g(x_n)df(x_n))\, x_n \to x\} \\
&= \mathrm{co}\{f(x)\lim dg(x_n) + g(x)\lim df(x_n),\, x_n \to x\} \\
&\subset f(x)\partial g(x) + g(x)\partial f(x)
\end{aligned}$$

$\square$

Bibliography

[1] M. Aaibid and A. Sayah. A direct proof of the equivalence between the entropy solutions of conservation laws and viscosity solutions of Hamilton-Jacobi equations in one-space variable. *JIPAM. J. Inequal. Pure Appl. Math.*, 7(2):Article 64, 11 pp. (electronic), 2006.

[2] M.-C. Arnaud. Pseudographs and the Lax-Oleinik semi-group: a geometric and dynamical interpretation. *Nonlinearity*, 24(1):71–78, 2011.

[3] Marie-Claude Arnaud. On a theorem due to Birkhoff. *Geom. Funct. Anal.*, 20(6):1307–1316, 2010.

[4] V. I. Arnol'd. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1988.

[5] Vladimir I. Arnold. *Lectures on partial differential equations*. Springer-Verlag, New York, 2000.

[6] S. A. Barannikov. The framed Morse complex and its invariants. In *Singularities and bifurcations*, volume 21 of *Adv. Soviet Math.*, pages 93–115. Amer. Math. Soc., Providence, RI, 1994.

[7] M. Bardi and L. C. Evans. On Hopf's formulas for solutions of Hamilton-Jacobi equations. *Nonlinear Anal.*, 8(11):1373–1381, 1984.

[8] Guy Barles. *Solutions de viscosité des équations de Hamilton-Jacobi*, volume 17 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Paris, 1994.

[9] Patrick Bernard. The Lax-Oleinik semi-group: a Hamiltonian point of view. CANPDE crash-course, Edinburgh, 2011.

[10] Patrick Bernard and Joana Oliveira Dos Santos. A geometric definition of the Aubry-Mather set. *J. Topol. Anal.*, 2(3):385–393, 2010.

[11] Patrick Bernard and Joana Oliveira dos Santos. A geometric definition of the Mañé-Mather set and a theorem of Marie-Claude Arnaud. *Math. Proc. Cambridge Philos. Soc.*, 152(1):167–178, 2012.

[12] Olga Bernardi and Franco Cardin. On C^0-variational solutions for Hamilton-Jacobi equations. *Discrete Contin. Dyn. Syst.*, 31(2):385–406, 2011.

[13] Marco Brunella. On a theorem of Sikorav. *Enseign. Math. (2)*, 37(1-2):83–87, 1991.

[14] Gianmarco Capitanio. Caractérisation géométrique des solutions de minimax pour l'équation de Hamilton-Jacobi. *Enseign. Math. (2)*, 49(1-2):3–34, 2003.

[15] Franco Cardin and Claude Viterbo. Commuting Hamiltonians and Hamilton-Jacobi multi-time equations. *Duke Math. J.*, 144(2):235–284, 2008.

[16] Kung Ching Chang. Variational methods for nondifferentiable functionals and their applications to partial differential equations. *J. Math. Anal. Appl.*, 80(1):102–129, 1981.

[17] Marc Chaperon. Generating maps in dynamics.

[18] Marc Chaperon. Une idée du type "géodésiques brisées" pour les systèmes hamiltoniens. *C. R. Acad. Sci. Paris Sér. I Math.*, 298(13):293–296, 1984.

[19] Marc Chaperon. An old-fashioned method in the calculus of variations. *P. H. Rabinowitz, A. Ambrosetti, I. Ekeland and E.J. Zehnder, Periodic solutions of Hamiltonian systems and related topics, Reidel Publishing Company*, pages 93–98, 1987.

[20] Marc Chaperon. Familles génératrices. Cours donné à l'école d'été Erasmus de Samos, http://www.math.jussieu.fr/~chaperon/prepublications.html, 1990.

[21] Marc Chaperon. Lois de conservation et géométrie symplectique. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(4):345–348, 1991.

[22] Marc Chaperon. *Calcul différentiel et calcul intégral,* 3^{e} *année.* Dunod, Paris, 2008.

[23] Yu. V. Chekanov. Critical points of quasifunctions, and generating families of Legendrian manifolds. *Funktsional. Anal. i Prilozhen.*, 30(2):56–69, 96, 1996.

[24] Alain Chenciner. Aspects géométriques de l'études des chocs dans les lois de conservation.

[25] Frank H. Clarke. Generalized gradients and applications. *Trans. Amer. Math. Soc.*, 205:247–262, 1975.

[26] Frank H. Clarke. A new approach to Lagrange multipliers. *Math. Oper. Res.*, 1(2):165–174, 1976.

[27] Frank H. Clarke. *Optimization and nonsmooth analysis.* Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons Inc., New York, 1983. A Wiley-Interscience Publication.

[28] M. G. Crandall, L. C. Evans, and P.-L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Trans. Amer. Math. Soc.*, 282(2):487–502, 1984.

[29] Michael G. Crandall and Pierre-Louis Lions. Remarks on the existence and uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations. *Illinois J. Math.*, 31(4):665–688, 1987.

[30] Constantine M. Dafermos. Polygonal approximations of solutions of the initial value problem for a conservation law. *J. Math. Anal. Appl.*, 38:33–41, 1972.

[31] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, second edition, 2005.

[32] Ya. M. Eliashberg. A theorem on the structure of wave fronts and its application in symplectic topology. *Funktsional. Anal. i Prilozhen.*, 21(3):65–72, 96, 1987.

[33] Albert Fathi. Weak KAM Theorem in Lagrangian Dynamics. book to appear, Cambridge University Press.

[34] Albert Fathi and Ezequiel Maderna. Weak KAM theorem on non compact manifolds. *NoDEA Nonlinear Differential Equations Appl.*, 14(1-2):1–27, 2007.

[35] Wendell Helms Fleming and H Mete Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer, New York, 2006.

[36] Laudenbach François. Homology de Morse dans la perspective de l'homologie de Floer. Mini-cours dans le cadre de la rencontre GIRAGA XIII,Yaoundé, 2010.

[37] Laudenbach François. Personal communication. 2011.

[38] Dmitry Fuchs and Dan Rutherford. Generating families and Legendrian contact homology in the standard contact space. *J. Topol.*, 4(1):190–226, 2011.

[39] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[40] Helge Holden and Nils Henrik Risebro. *Front tracking for hyperbolic conservation laws*, volume 152 of *Applied Mathematical Sciences*. Springer, New York, 2011. First softcover corrected printing of the 2002 original.

[41] Eberhard Hopf. Generalized solutions of non-linear equations of first order. *J. Math. Mech.*, 14:951–973, 1965.

[42] Shyūichi Izumiya and Georgios T. Kossioris. Semi-local classification of geometric singularities for Hamilton-Jacobi equations. *J. Differential Equations*, 118(1):166–193, 1995.

[43] Shyuichi Izumiya and Georgios T. Kossioris. Formation of singularities for viscosity solutions of Hamilton-Jacobi equations. In *Singularities and differential equations (Warsaw, 1993)*, volume 33 of *Banach Center Publ.*, pages 127–148. Polish Acad. Sci. Inst. Math., Warsaw, 1996.

[44] E. R. Jakobsen, K. H. Karlsen, and N. H. Risebro. On the convergence rate of operator splitting for weakly coupled systems of Hamilton-Jacobi equations. In *Hyperbolic problems: theory, numerics, applications, Vol. I, II (Magdeburg, 2000)*, volume 141 of *Internat. Ser. Numer. Math., 140*, pages 553–562. Birkhäuser, Basel, 2001.

[45] Tatiana Joukovskaia. *Singularité de minimax et solutions faibles d'équations aux dérivées partielles.* PhD thesis, Université de Paris 7, 1993.

[46] G. T. Kossioris. Formation of singularities for viscosity solutions of Hamilton-Jacobi equations in one space variable. *Comm. Partial Differential Equations*, 18(5-6):747–770, 1993.

[47] S. N. Kruzkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[48] Peter D. Lax. *Hyperbolic systems of conservation laws and the mathematical theory of shock waves.* Society for Industrial and Applied Mathematics, Philadelphia, Pa., 1973. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, No. 11.

[49] P.-L. Lions and J.-C. Rochet. Hopf formula and multitime Hamilton-Jacobi equations. *Proc. Amer. Math. Soc.*, 96(1):79–84, 1986.

[50] Pierre-Louis Lions. *Generalized solutions of Hamilton-Jacobi equations*, volume 69 of *Research Notes in Mathematics.* Pitman (Advanced Publishing Program), Boston, Mass., 1982.

[51] David McCaffrey. Graph selectors and viscosity solutions on Lagrangian manifolds. *ESAIM Control Optim. Calc. Var.*, 12(4):795–815 (electronic), 2006.

[52] John Milnor. *Lectures on the h-cobordism theorem.* Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

[53] Gabriel P. Paternain, Leonid Polterovich, and Karl Friedrich Siburg. Boundary rigidity for Lagrangian submanifolds, non-removable intersections, and Aubry-Mather theory. *Mosc. Math. J.*, 3(2):593–619, 745, 2003. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday.

[54] R. Tyrrell Rockafellar and Roger J.-B. Wets. *Variational analysis*, volume 317 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1998.

[55] Joshua M. Sabloff and Lisa Traynor. Obstructions to the existence and squeezing of Lagrangian cobordisms. *J. Topol. Anal.*, 2(2):203–232, 2010.

[56] Matthias Schwarz. *Morse Homology.* Birkhäuser, Boston, 1993.

[57] Jean-Claude Sikorav. Problèmes d'intersections et de points fixes en géométrie hamiltonienne. *Comment. Math. Helv.*, 62:345–348, 1987.

[58] Panagiotis E. Souganidis. Approximation schemes for viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 59(1):1–43, 1985.

[59] Panagiotis E. Souganidis. Existence of viscosity solutions of Hamilton-Jacobi equations. *J. Differential Equations*, 56(3):345–390, 1985.

[60] David Théret. A complete proof of Viterbo's uniqueness theorem on generating functions. *Topology Appl.*, 96(3):249–266, 1999.

[61] J. van Tiel. *Convex analysis: an introductory text.* Wiley, 1984.

[62] Claude Viterbo. Symplectic topology as the geometry of generating functions. *Math. Ann.*, 292(4):685–710, 1992.

[63] Claude Viterbo. Solutions d'équations d'Hamilton-Jacobi et géométrie symplectique. Séminaire sur les équations aux dérivées partielles, Palaiseau, 1996.

[64] Claude Viterbo. Symplectic topology and Hamilton-Jacobi equations. In *Morse theoretic methods in nonlinear analysis and in symplectic topology*, volume 217 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 439–459. Springer, Dordrecht, 2006.

[65] Claude Viterbo and Alberto Ottolenghi. Variational solutions of Hamilton-Jacobi equations. avalable from http://math.polytechnique.fr/ viterbo, 1995.